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BEST PROXIMITY POINT THEOREM FOR MAPPINGS WITH A CONTRACTIVE ITERATE ON ℙ SETS

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ABSTRACT: *There are many generalizations of the Banach contraction principle. In this paper we prove a result that combines three of them: best proximity points of a cyclic map in a complete metric space with the UC property, the contraction condition holding for a subset of all elements, and the mapping being with a contractive iterate at a point. We illustrate this result by proving Eldred and Veermani's classic theorem as a corollary to ours and illustrate an application of it by an example.*

KEYWORDS: *Best proximity point, ℙ set, UC property*

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1 Introduction

The Banach contraction principle's [1] importance hardly needs an introduction. Ever since its formulation, it has been used both in theory and application for reasoning about equations of the form $Tx = x$. Such success has prompted the development of many generalizations.

One generalization addresses problems, where the cyclic map $T : A \to B$ and $T : B \to A$ does not have a fixed point. In that case, conditions for the existence and uniqueness of a suitable closest element to Tx, called a best proximity point, are sought [5]. That turns the fixed point problem into an optimization problem, often in the setting of uniformly convex Banach spaces, $\min_{x\in A\cup B}\{\|x-Tx\|\}$. An investigation for suitable conditions on the sets A,B in complete metric spaces for existence and uniqueness of best proximity points has first been conducted in [20]. The theory of best proximity points has its share of applications, some being in differential equations [12] and game theory [11].

Another generalization consists of looking for fixed points when the contraction condition holds only for some elements [16]. There are a multitude of fixed point theorems in partially ordered metric spaces, e.g., results for nonlinear matrix equations [16], for coupled fixed points [2], for general orderings defined via a ℙ set [13], et cetera.

Yet another type of generalization is found in [9, 10, 17]. In this case the mapping has a contraction iterate at a point, that is, the contraction condition is valid only after n repeated applications of T on x, n being different for different x. This leads to mappings that could exhibit highly oscillatory behavior and yet converge to a fixed point by the simple Picard iteration.

The goal of this paper is to unite these three generalizations into one result, that is, to find existence and uniqueness conditions for best proximity points for a subset of $A\times B$, where the contraction condition requires a varying amount of applications of T, depending on the element on which T is being applied.

2 Preliminaries

In what follows, we will use the notations: $\mathbb{N}$ for the set of natural numbers, $\mathbb{R}$ for the set of real numbers, (X,ρ) is a metric space and $A,B\subset X$. We will denote by $\operatorname{dist}(A,B)=\inf\{\rho(a,b) : A\in A, b\in B\}$ the distance between the sets A and B.

The theory of best proximity points obtained by cyclic maps was initiated in [5].

Definition 1. [5] Let A and B be nonempty subsets of a metric space X. A map $T : A\cup B \to A\cup B$

is a cyclic map if

$$T(A) \subset B \text{ and } T(B) \subset A.$$

Definition 2. [5] Let A and B be nonempty subsets of a metric space X, such that $A \cup B = \emptyset$. We say that $x \in A \cup B$ is a best proximity point if

$$d(x, Tx) = \operatorname{dist}(A, B).$$

Definition 3. ([5]) Let (X, ρ) be a metric space, A and B be subsets of X. We say that the map $T : A \cup B \to A \cup B$ is a cyclic contraction map, if it is a cyclic map and satisfies the inequality

$$\rho(Tx, Ty) \leq \alpha \rho(x, y) + (1 - \alpha)\operatorname{dist}(A, B)$$

for some $\alpha \in (0, 1)$ and every $x \in A$, $y \in B$.

In order to get results about existence and uniqueness, the notion of a uniform convexity of the underlying Banach space is crucial.

Definition 4. ([3, 6]) Let $(X, \|\cdot\|)$ be a Banach space. For every $\varepsilon \in (0, 2]$ we define the modulus of convexity of $\|\cdot\|$ by

$$\delta_{\|\cdot\|}(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X, \|x - y\| \geq \varepsilon\right\}.$$

The norm is called uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. The space $(X, \|\cdot\|)$ is then called uniformly convex Banach space.

Whenever the underlying space is a Banach space $(X, \|\cdot\|)$ we will consider the metric to be the one induced by the norm, i.e., $\rho(x, y) = \|x - y\|$.

Theorem 1. ([5]) Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space $(X, \|\cdot\|)$. Suppose $T : A \cup B \to A \cup B$ be a cyclic contraction map, then there exists a unique best proximity point x of T in A.

Later on, it has been observed [20] that the uniform convexity of the underlying Banach space is too restrictive an assumption. Several notions that can replace uniform convexity have been considered in [8, 18, 19, 20]. Connections between the mentioned notions, uniform convexity and some generalizations have been presented in [21].

Definition 5. [13] Let (X, d) be a metric space. We say that two sequences $\{x_n\}$, $\{y_n\} \subset X$ are Cauchy equivalent if $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Definition 6. [20] Let (X, d) be a metric space and $A, B \subset X$ be non-empty. We say that the ordered pair (A, B) satisfies the property UC if for any sequences $\{x_n\}, \{z_n\} \subset A$ and $\{y_n\} \subset B$ such that $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(z_n, y_n) = \operatorname{dist}(A, B)$, the sequences $\{x_n\}$ and $\{z_n\}$ are Cauchy equivalent.

Theorem 2. ([20]) Let A and B be nonempty closed subsets of a complete metric space (X, ρ), such that the ordered pairs (A, B) satisfy the property UC. Let $T : A \cup B \to A \cup B$ be a cyclic map and there exist $k \in [0, 1)$, so that the inequality

$$\rho(Tx, Ty) \leq k \max\{\rho(x, y), \rho(x, Tx), \rho(y, Ty)\} + (1 - k)\operatorname{dist}(A, B)$$

holds for all $x \in A$ and $y \in B$.

Then there is a unique best proximity point x of T in A, the sequence of successive iterations $\{T^{2n}x_0\}_{n=1}^{\infty}$ converges to x for any initial guess $x_0 \in A$. There is at least one best proximity point $y \in B$ of T in B. Moreover, the best proximity point $y \in B$ of T in B is unique, provided that the ordered pair (B,A) has the *UC* property.

The first result about fixed points in partially ordered metric spaces is in [16], where the contractive condition $\rho(Tx,Ty) \leq k\rho(x,y)$ is weakened by assuming that it is satisfied only for $x \preccurlyeq y$.

Theorem 3. ([16]) Let $(X,d,\preccurlyeq)$ be a partially ordered complete metric spaces and $f : X \to X$ be a continuous, monotone (i.e., either order preserving or order reversing) map, such that there is $k \in [0,1)$ so that the inequality

$$d(Tx,Ty) \leq kd(x,y)$$

holds true for arbitrary $x,y \in X$, satisfying $x \succcurlyeq y$. A fixed point $\xi \in X$ of T exists if there is $x_0 \in X$ such that either $x_0 \preccurlyeq fx_0$ or $x_0 \succcurlyeq fx_0$.

The fixed point ξ will be unique if each pair of elements $x,y \in X$ possesses a lower bound or an upper bound.

The partial order in the underlying metric space (X,ρ) assures that only for some elements from X the contractive condition is satisfied. This idea has later been generalized in [13] by replacing the notion of partial order with $\mathbb{P}$ sets in $X \times X$. The main purpose of the sequence of articles [13, 14, 15] is the investigation of coupled fixed points, where the $\mathbb{P}$ sets were subsets of X^4. The ideas about coupled fixed points from these articles have been developed for fixed points in [7].

Definition 7. ([7]) Let (X,d) be a metric space, $F : X \to X$ and $\mathbb{P} \subset X \times X$. The set $\mathbb{P}$ is called F-regular provided that $(x,F(x)) \in \mathbb{P}$ for every sequence $(x_n,F(x_n)), n \in \mathbb{N}$ in $\mathbb{P}$ such that $\lim_{n\to\infty} x_n = x$.

Definition 8. ([7]) Let (X,d) be a metric space. $F : X \to X$ be a map and $\mathbb{P} \subset X \times X$ be F-regular. Let $V = \{x \in X : (x,F(x)) \in \mathbb{P}\}$ we say that the function $T : X \to \mathbb{R} \cup \{+\infty\}$ is lower l.s.c. on V (u.s.c on V) if at any $x_0, x_n \in V$, such that $\lim_{n\to\infty} x_n = x_0$, there holds $\liminf\limits_{n\to\infty} T(x_n) \geq T(x_0) (\limsup\limits_{n\to\infty} T(x_n) \leq T(x_0))$. Additionally, if $T(x) \not\equiv +\infty$ for $x \in V$, it is called a proper function on V.

Definition 9. ([7]) Let $(X,d,\preccurlyeq)$ be a partially ordered metric space. We say that a map $f : X \to X$ is a l.s.c. (u.s.c.) with respect to $\preccurlyeq$ if any sequence x_n, such that $x_n \asymp x_0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x_0$ there holds $\liminf\limits_{n\to\infty} f(x_n) \succcurlyeq f(x_0) (\limsup\limits_{n\to\infty} f(x_n) \preccurlyeq f(x_0))$.

Theorem 4. ([7]) Let (X,d) be a complete metric space, $F : X \to X$ be a map, $\mathbb{P} \subset X \times X$ and let $\mathbb{P}$ be F-regular. Let $V = \{x \in X : (x,F(x)) \in \mathbb{P}\}$ and $T : X \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c. bounded from below function on V.

Let $\varepsilon > 0$ be arbitrary but fixed. Let $u_0 \in V$ be such that

$$T(u_0) \leq \inf_{v\in V} T(v) + \varepsilon. \qquad (1)$$

Then there exists $x \in V$ such that

(i) $T(x) \leq \inf_{v \in V} T(v) + \varepsilon$

(ii) $d(x, u_0) \leq 1$

(iii) for all $w \in V, w \neq x$ there holds $T(w) > T(x) - \varepsilon d(w, x)$.

Definition 10. [15] Let (X,d) be a metric space, $\mathbb{P} \subset X \times X$ and $F : X \to X$ be a mapping. $\mathbb{P}$ is called F-closed if

$$(x,y) \in \mathbb{P} \Rightarrow (F(x), F(y)) \in \mathbb{P}.$$

The next examples are well known [13].

Example 1. Let $(X, d, \preccurlyeq)$ be a partially ordered metric space. Let $F : X \to X$ be an increasing function, i.e., $F(x) \preccurlyeq F(y)$, provided that $x \preccurlyeq y$. Then the set $\mathbb{P} = \{(x,y) \in X \times X : x \preccurlyeq y\}$ is F-closed.

Example 2. Let $(X, d, \preccurlyeq)$ be a partially ordered metric space. For $F : X \to X$ let $F(x)$ be comparable with $F(y)$, i.e. $F(x) \asymp F(y)$. Then the set $\mathbb{P} = \{(x,y) \in X \times X : x \asymp y\}$ is F-closed.

The next theorem is a combination of the Banach fixed point theorem and the results of [13, 16].

Theorem 5. ([7]) Let $(X, d, \preccurlyeq)$ be a partially ordered metric space, $\mathbb{P} = \{(x,y) \in X \times X : x \succcurlyeq y\}, F : X \to X$ be a mapping l.s.c. with respect to $\preccurlyeq$ and $V = \{x \in X : (x, F(x)) \in \mathbb{P}\}$. Suppose that $\mathbb{P}$ is F-closed, $V \neq \emptyset$ and the function $x \mapsto d(x, F(x))$ is l.s.c. on V.

If there exists $\alpha \in [0,1)$ such that

$$d(F(x), F(y)) \leq \alpha d(x,y)$$

for all $(x,y) \in \mathbb{P}$, then F has a fixed point in X.

If, additionally, for every pair x, y of fixed points there exists $z \in X$ such that one of the inclusions $(x,z), (z,y) \in \mathbb{P}, (x,z), (y,z) \in \mathbb{P}$ or $(z,x), (z,y) \in \mathbb{P}$ holds, then the fixed point is unique.

A different approach has been proposed in [17].

Theorem 6. ([17]) Let X be a Banach space, and $T : X \to X$ a continuous mapping satisfying the condition: there exists a constant $\alpha \in (0,1)$ such that for each $x \in X$, there is a positive integer $n(x)$ such that for all $y \in X$

$$\rho(T^{n(x)}y, T^{n(x)}x) \leq \alpha \rho(y,x).$$

Then T has a unique fixed point z and $\lim_{s \to \infty} T^s x = z$ for each $x \in X$.

Later, the maps introduced in [17] have been named maps iterated at a point. We would like to note that generalizations of the results from [17] have been presented in [10, 9]

3 Main Result

Inspired by [7, 14], we give the following definitions:

Definition 11. Let (X,d) be a metric space, $A,B \subset X$ and $\mathbb{P} \subset A \times B$, $A \cap B = \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence such that $x_{2n} \in A$ and $x_{2n+1} \in B$. The triple $(A \cup B, d, \mathbb{P})$ is said to be:

- cyclically e-$\mathbb{P}$-regular if for any sequence $\{x_{2n}\}$, convergent to x^*, such that $(x_{2n}, x_{2n+1}) \in \mathbb{P}$ for all $n \in \mathbb{N}$, there holds $(x^*, x_{2n+1}) \in \mathbb{P}$ for all $n \in \mathbb{N} \cup \{0\}$
- cyclically o-$\mathbb{P}$-regular if for any sequence $\{x_{2n+1}\}$, convergent to x^*, such that $(x_{2n}, x_{2n+1}) \in \mathbb{P}$ for all $n \in \mathbb{N}$, there holds $(x_{2n}, x^*) \in \mathbb{P}$, for all $n \in \mathbb{N} \cup \{0\}$.

Definition 12. Let X be a non-empty set, $A,B \subset X$, $\mathbb{P} \subset A \times B$ and let $T : A \cup B \to A \cup B$ be a cyclic map. We say that $\mathbb{P}$ is cyclically T-closed if

$$(x,y) \in \mathbb{P} \Rightarrow (Ty, Tx) \in \mathbb{P}$$

Definition 13. We say that $\mathbb{P}$ has the cyclically transitive property if from $(x,y),(z,y),(z,u) \in \mathbb{P}$ it follows that $(x,u) \in \mathbb{P}$

We will first prove a technical lemma.

Lemma 1. Let (X,d) be a metric space, $A,B \subset X$ be nonempty such that $A \cap B = \emptyset$, $\mathbb{P} \subset A \times B$, $T : A \cup B \to A \cup B$ be a cyclic map, $a \in A, b \in B$ and there exists $k \in [0,1)$ such that for all $x \in A \cup B$ there is $n(x) \equiv 1 \pmod 2 \in \mathbb{N}$, such that for all $y \in A \cup B$, where (x,y) or $(y,x) \in \mathbb{P}$, we have

$$d(T^{n(x)}(x), T^{n(x)}(y)) \leq kd(x,y) + (1-k)\mathrm{dist}(A,B).$$

Then if $(T^{2n}a, T^{2m}b) \in \mathbb{P}$ for $n,m \in \mathbb{N} \cup \{0\}$, then $\sup_{n \in \mathbb{N} \cup \{0\}} \{d(T^{2n}b, a)\} < +\infty$.

Proof. Let $l = \max\{d(T^c a, T^d a), c = 0,1, d = 0,1,2\ldots, \max(n(a), n(b))\}$. There exist $p,q,s \in \mathbb{N}$ such that $2n+1 = pn(a) + qn(b) + s, 0 \leq p - q \leq 1, 0 \leq s < \max\{n(a), n(b)\}$. Then

$$\begin{aligned}
d(T^{2n}b, a) &\leq d(T^{2n}b, T^{n(Ta)}Ta) + d(T^{n(Ta)}Ta, a) \\
&\leq kd(T^{2n-n(Ta)}b, Ta) + (1-k)\mathrm{dist}(A,B) + l \\
&\leq k(d(T^{2n-n(Ta)}b, T^{n(a)}a) + d(T^{n(a)}a, a)) + (1-k)\mathrm{dist}(A,B) + l \\
&\leq k^2(d(T^{2n-n(Ta)-n(a)}b, a)) + (1-k^2)\mathrm{dist}(A,B) + l + kl \\
&\leq \ldots \\
&\leq k^{p+q}d(T^s b, T^c a) + (1-k^{p+q})\mathrm{dist}(A,B) + l\textstyle\sum_{i=0}^{s+p-1} k^i \\
&\leq (1-k^{p+q})\mathrm{dist}(A,B) + l\textstyle\sum_{i=0}^{s+p} k^i \leq \mathrm{dist}(A,B) + \frac{l}{1-k} < +\infty.
\end{aligned}$$

Therefore, $\sup_{n \in \mathbb{N} \cup \{0\}} \{d(T^{2n}b, a)\}$ is finite. □

Now we are ready to proceed with the main result.

Theorem 7. Let (X,d) be a complete metric space, $A,B \subset X$ be nonempty such that $A \cap B = \emptyset$, the pair (A,B) have the *UC* property, $\mathbb{P} \subset A \times B$, $T : A \cup B \to A \cup B$ be a cyclic map and there hold

(i) $\mathbb{P}$ is cyclically T-closed and has the cyclically transitive property;

(ii) the triple $(A\cup B, d, \mathbb{P})$ is cyclically e-$\mathbb{P}$-regular;

(iii) there exists $x_0 \in A$ such that $(x_0, Tx_0) \in \mathbb{P}$;

(iv) there exists $k \in [0,1)$ such that for all $x \in A\cup B$ there is $n(x) \equiv 1 \pmod 2 \in \mathbb{N}$, such that for all $y \in A\cup B$, where (x,y) or $(y,x) \in \mathbb{P}$, we have

(2) $$d(T^{n(x)}(x), T^{n(x)}(y)) \le kd(x,y) + (1-k)\mathrm{dist}(A,B).$$

Then there exists a best proximity point x^* in A and for any arbitrarily chosen $x_0 \in A$, such that $(x_0, Tx_0) \in \mathbb{P}$ the iterated sequence $x_{2n} = T^{2n}x_0$ converges to a best proximity point. Furthermore, x^* is a fixed point of T^2. Moreover, there hold

(a) for any $x \in A$ such that $(x_0, Tx) \in \mathbb{P}$ or $(x, Tx_0) \in \mathbb{P}$, the sequences $x_{2n} = T^{2n}x_0$ and $u_{2n} = T^{2n}x$ are Cauchy equivalent and hence u_{2n} converges to the same point x^*;

(b) if $y^* \in A$ is a best proximity point and either $(x_0, Ty^*) \in \mathbb{P}$ or $(y^*, Tx_0) \in \mathbb{P}$ or there exists $z \in A$ so that either $(x_0, Tz), (y^*, Tz) \in \mathbb{P}$ or $(z, Tx_0), (z, Ty^*) \in \mathbb{P}$ then $y^* = x^*$;

(c) if additionally we suppose that for every $x, y \in A$ such that neither $(x, Ty) \in \mathbb{P}$ or $(y, Tx) \in \mathbb{P}$ there is $z \in A$ so that either $(x, Tz), (y, Tz) \in \mathbb{P}$ or $(z, Tx), (z, Ty) \in \mathbb{P}$ then x^* is the unique proximity point.

Proof. By assumption, we have $x_0 \in A$ such that $(x_0, Tx_0) \in \mathbb{P}$.

By (i) it is clear that

(3) $$(T^{2n}x_0, T^{2m+1}x_0) \text{ for } n, m \in \mathbb{N}\cup\{0\}.$$

Setting $a = x_0$ and $b = Tx_0$ in Lemma 1, we get that $r = \sup_{n\in\mathbb{N}\cup\{0\}}\{d(T^{2n+1}x_0, x_0)\}$ is finite.

Let us define the sequence z_n thus: $z_0 = x_0, z_1 = T^{m_0}z_0$ where $m_0 = n(z_0)$, $z_2 = T^{m_1}z_1$, where $m_1 = n(z_1)$ and in general $z_{n+1} = T^{m_n}z_n, m_n = n(z_n)$. For the rest of the paper we will also use $x_{n+1} = Tx_n$. Let us denote $s_n = \sum_{i=0}^{n} m_i$ and note that s_{2n+1} is even, whereas s_{2n} is odd for $n \in \mathbb{N}\cup\{0\}$. From the chain of inequalities

$$\begin{aligned} d(z_{n+1}, z_n) &= d(T^{s_n}x_0, T^{s_{n-1}}x_0) = d(T^{m_{n-1}}T^{m_n+s_{n-2}}x_0, T^{m_{n-1}}T^{s_{n-2}}x_0) \\ &\le kd(T^{m_n+s_{n-2}}x_0, T^{s_{n-2}}x_0) + (1-k)\mathrm{dist}(A,B) \le \dots \\ &\le k^n d(T^{m_n}x_0, x_0) + (1-k^n)\mathrm{dist}(A,B) \le k^n r + (1-k^n)\mathrm{dist}(A,B) \end{aligned}$$

we conclude that $d(z_{n+1}, z_n) \to \mathrm{dist}(A,B)$.

We will prove that z_{2n} is a Cauchy sequence by way of contradiction. Let us suppose that z_{2n} is not a Cauchy sequence, i.e. there exists $\varepsilon > 0$ such that for every $2N$, $N \in \mathbb{N}$ there exist $l_1(N), l_2(N) \ge 2N, l_1, l_2 \in 2\mathbb{N}$ such that $d(z_{l_1}, z_{l_2}) > \varepsilon$, which would mean, taking into account that $\lim_{N\to\infty} l_1(N) = \lim_{N\to\infty} l_2(N) = \infty$, that

(4) $$\lim_{N\to\infty} d(z_{l_1}, z_{l_2}) \ne 0$$

Let us observe the inequalities

$$\begin{aligned} d(z_{l_1}, z_{2N-1}) &\leq kd(z_{l_1-1}, z_{2N-2}) + (1-k)\text{dist}(A,B) \\ &\leq \dots \\ &\leq k^{2N-1}d(z_{l_1-(2N-1)}, z_0) + (1-k^{2N-1})\text{dist}(A,B) \\ &\leq k^{2N-1}r + (1-k^{2N-1})\text{dist}(A,B) \to \text{dist}(A,B). \end{aligned}$$

We see that $d(z_{l_1}, z_{2N-1}) \to \text{dist}(A,B)$. Similarly, we can show that $d(z_{l_2}, z_{2N-1}) \to \text{dist}(A,B)$. From the UC property of (A,B) we conclude that $d(z_{l_1}, z_{l_2}) \to 0$, contradicting (4). Thus, z_{2n} is a Cauchy sequence, and there exists $x^* \in A$, which is a limit of the sequence $\{z_{2n}\}$.

Next, we will show that $\{z_{2n}\}$ and $\{x_{2n}\}$ are Cauchy equivalent. Let us choose $m \in \mathbb{N}$ such that $2m(n) \geq s_{2n}$. We observe that $\lim_{n\to\infty} m(n) = \infty$ and

$$\begin{aligned} d(x_{2m}, z_{2n+1}) &= d(T^{2m}x_0, T^{s_{2n}}x_0) = d(T^{m_{2n}}T^{2m-m_{2n}}x_0, T^{m_{2n}}T^{s_{2n-1}}x_0) \\ &\leq kd(T^{2m-m_{2n}}x_0, T^{s_{2n-1}}x_0) + (1-k)\text{dist}(A,B) \leq \dots \\ &\leq k^{2n+1}d(T^{2m-s_{2n}}x_0, x_0) + (1-k^{2n+1})\text{dist}(A,B) \\ &\leq k^{2n+1}r + (1-k^{2n+1})\text{dist}(A,B). \end{aligned}$$

If follows that $d(x_{2m}, z_{2n+1}) \to \text{dist}(A,B)$ and from the UC property we get that $x_{2m} \to x^*$.

We will establish that $d(x_{2n}, x_{2n+1}) \to \text{dist}(A,B)$. We can find $p(n), i(n) \in \mathbb{N} \cup \{0\}$ such that $2n = s_{2i} + p, 0 \leq p < s_{2i+1} - s_{2i}$. Observing that $2i(n) \to \infty$ as $n \to \infty$ and $d(x_0, T^{2a}x_0) \leq M$ for $a \in \mathbb{N}$ and $M > 0$, since the sequence $\{x_{2n}\}$ is convergent, we get

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\leq d(x_{2n}, z_{2i}) + d(z_{2i}, x_{2n+1}) \\ &\leq d(x_{2n}, z_{2i}) + k^{2i}d(x_0, T^{p+1}x_0) + (1-k^{2i})\text{dist}(A,B) \\ &\leq d(x_{2n}, z_{2i}) + k^{2i}\max\{r, M\} + (1-k^{2i})\text{dist}(A,B). \end{aligned}$$

Thus, we have shown that $d(x_{2n}, x_{2n+1}) \to \text{dist}(A,B)$.

To establish that x^* is a fixed point of some power of T, we first observe that the triple $(A \cup B, d, \mathbb{P})$ is cyclically e-$\mathbb{P}$-regular by assumption. Since $(x^*, x_{2m+1}) \in \mathbb{P}$ for $m \in \mathbb{N} \cup \{0\}$ and $\mathbb{P}$ being cyclically T-closed we get that $(x_{2m}, Tx^*) \in \mathbb{P}$ for all $m \in \mathbb{N}$. Furthermore, since $(x^*, x_{2n+1}), (x_{2n}, x_{2n+1}), (x_{2n}, Tx^*) \in \mathbb{P}$, then by the cyclically transitive property of $\mathbb{P}$ we get that $(x^*, Tx^*) \in \mathbb{P}$. It is easily proven true that $(T^a x^*, T^b x^*) \in \mathbb{P}$ for $a, b \in \mathbb{N} \cup \{0\}, a - b \equiv 1 \pmod 2$.

Let us use $v = n(x^*) + n(T^{n(x^*)}x^*)$ for simplicity. By the inequalities

$$d(x_{2n+1}, x^*) \leq d(x_{2n+1}, x_{2n}) + d(x_{2n}, x^*) \to \text{dist}(A,B) \tag{5}$$

and for $2n+1 \geq n(x^*) + n(T^{n(x^*)})$

$$d(x_{2n+1}, T^v x^*) \leq k^2 d(x_{2n+1-v}, x^*) + (1-k^2)d(A,B) \to d(A,B)$$

we get that $d(x_{2n+1}, T^v x^*) \to \text{dist}(A,B)$. By the UC property we get that $\lim_{n\to\infty} d(x^*, T^v x^*) = 0$ and we conclude that $x^* = T^v x^*$.

Next we will show that $d(x^*, Tx^*) = \text{dist}(A,B)$. Observing that by applying (2) twice, we get

$$\begin{aligned} d(x^*, Tx^*) &= d(T^v x^*, T^{v+1}x^*) \\ &\leq k^2 d(x^*, Tx^*) + (1-k^2)\text{dist}(A,B), \end{aligned}$$

from which it follows that $(1-k^2)d(x^*,Tx^*) \le (1-k^2)\mathrm{dist}(A,B)$ and using

$$\mathrm{dist}(A,B) \le d(x^*,Tx^*) \le \mathrm{dist}(A,B)$$

we conclude that $d(x^*,Tx^*) = \mathrm{dist}(A,B)$.

What is left to do is to show that $T^2x^* = x^*$. In order to do that, we will prove that $(x_{2m},T^{2n+1}x^*) \in \mathbb{P}, m,n \in \mathbb{N}\cup\{0\}$ by induction. We have shown that $(x_{2m},Tx^*) \in \mathbb{P}, m \in \mathbb{N}$. By $(x_0,x_1),(x_2,x_1),(x_2,Tx^*) \in \mathbb{P}$, it follows that $(x_{2m},Tx^*) \in \mathbb{P}, m \in \mathbb{N}\cup\{0\}$. If $(x_{2m},T^{2l+1}x^*) \in \mathbb{P}$, $m \in \mathbb{N}\cup\{0\}$ for some $l \in \mathbb{N}\cup\{0\}$, then by using $(x_{2m-2},T^{2l+1}x^*) \in \mathbb{P}$ and $\mathbb{P}$ being cyclically T-closed, it follows that $(x_{2m},T^{2l+3}x^*) \in \mathbb{P}, m \in \mathbb{N}$, and via (3), $(x_0,x_{2m+1}),(x_{2m},x_{2m+1})$, $(x_{2m},T^{2l+3}x^*) \in \mathbb{P}$ and the cyclically transitive property of $\mathbb{P}$, we conclude that $(x_{2m},T^{2l+3}x^*) \in \mathbb{P}$, $m \in \mathbb{N}\cup\{0\}$. Therefore, $(x_{2m},T^{2n+1}x^*) \in \mathbb{P}, m,n \in \mathbb{N}\cup\{0\}$. Setting $a = x_0$ and $b = Tx^*$ in Lemma 1, we get that $r^* = \sup_{n\in\mathbb{N}\cup\{0\}}\{d(T^{2n+1}x^*,x_0)\}$ is finite. Let us choose $q,n \in \mathbb{N}$ such that $q(n)v \ge s_{2n}$. Due to s_{2n} being odd and v being even, $2+qv-s_{2n}$ is odd, and by the inequalities

$$\begin{array}{rcl} d(T^2x^*,x_{s_{2n}}) & = & d(T^{2+qv}x^*,T^{s_{2n}}x_0) \\ & \le & k^{2n+1}d(T^{2+qv-s_{2n}}x^*,x_0)+(1-k^{2n+1})\mathrm{dist}(A,B) \\ & \le & k^{2n+1}r^*+1-k^{2n+1}\mathrm{dist}(A,B) \to \mathrm{dist}(A,B) \end{array}$$

and (5), it follows from the UC property of (A,B) that $T^2x^* = x^*$.

(a)Without loss of generality, let us assume that $(x,Tx_0) \in \mathbb{P}$ and define $u_n = T^nx$. Let us find $m,n \in \mathbb{N}$ such that $s_{2n+1} \ge 2m(n) \ge s_{2n}$ and observe that

$$\begin{array}{rcl} d(u_{2n},x_{s_{2n}}) & = & d(T^{2m}x,T^{s_{2n}}x_0) \\ & \le & k^{2n+1}d(T^{2m-s_{2n}}x,x_0)+(1-k^{2n+1})\mathrm{dist}(A,B) \\ & \le & k^{2n+1}\max\{d(T^ix,x_0) : i=0,1,2,\ldots m_{2n+1}\}+(1-k^{2n+1})\mathrm{dist}(A,B) \to \mathrm{dist}(A,B) \end{array}$$

and by the UC property of (A,B) we get that $\{u_{2n}\}$ and $\{x_{2n}\}$ are Cauchy equivalent.

(b) If we have (x_0,Ty^*) or $(y^*,Tx_0) \in \mathbb{P}$ then by (a) we conclude that $x^* = y^*$.

In order to finish the proof of Theorem 7, we will state and prove the following

Proposition 1. Let (X,d) be a complete metric space, $A,B \subset X$ be nonempty such that $A\cap B = \emptyset$, the pair (A,B) has the *UC* property, $\mathbb{P} \subset A\times B$, $T : A\cup B \to A\cup B$ be a cyclic map and there hold (i), (ii) and (iv). Then if we have a sequence $u_{n+1} = Tu_n, u_0 \in A$ such that $u_{2n} \to u^*$, a best proximity point, and either (u_0,Tw_0) or $(w_0,Tu_0) \in \mathbb{P}$ for $w_{n+1} = Tw_n, w_0 \in A$ we have that the two sequences $\{u_{2n}\}$ and $\{w_{2n}\}$ are Cauchy equivalent.

Proof. This can be proven by replacing the powers m_i of $\{x_n\}$ needed for (iv) by powers, needed for the sequence $\{u_n\}$. □

Without loss of generality, let us have $z \in A$ such that $(x_0,Tz),(y^*,Tz) \in \mathbb{P}$. By $(x_0,Tz) \in \mathbb{P}$ and (a) we find that $T^{2n}z \to x^*$ and by (y^*,Tz) and Proposition 1 we get that $T^{2n}z \to y^*$. Therefore, $x^* = y^*$.

(c) Let us assume that there exists a different best proximity point $y^* \in A$. We will show that $y^* = x^*$. Without loss of generality, let us have $z \in A$ such that $(x^*,Tz),(y^*,Tz) \in \mathbb{P}$. Then by Proposition 1 we get that $T^{2n}z \to x^*$ and $T^{2n}z \to y^*$. Thus, $x^* = y^*$, showing that x^* is the unique best proximity point in A. □

4 Applications

We will show that the classic result in [5] is a corollary of Theorem 7.

Definition 14. [5] Let A and B be nonempty subsets of a metric space X. A map $T : A \cup B \to A \cup B$ is a cyclic contraction map if it satisfies:

1. $T(A) \subset B$ and $T(B) \subset A$;
2. for some $k \in (0,1)$ we have $d(Tx,Ty) \leq kd(x,y) + (1-k)\text{dist}(A,B)$ for all $x \in A, y \in B$.

Theorem 8. [5] Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space. Suppose $T : A \cup B \to A \cup B$ is a cyclic contraction map. Then there exists a unique best proximity point x^* in A (that is with $\|x^* - Tx^*\| = \text{dist}(A,B)$). Further, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point.

Proof. Much like in [5], if $A \cap B \neq \emptyset$, then the result follows from the Banach contraction principle. Therefore, let $A \cap B = \emptyset$. We will use Theorem 7 to prove the result.

It is clear that X is a complete metric space. That the pair (A,B) has the property UC is evident[20]. Due to T being a cyclic contraction, we get that by setting $\mathbb{P} = A \times B$ and $n(x) = 1$, $x \in A \cup B$, we fulfill condition (iv). Furthermore, by $\mathbb{P} = A \times B$, conditions (i), (ii), (iii) and there existing $z \in A$ such that $(x,Tz),(y,Tz) \in \mathbb{P}$ or $(z,Tx),(z,Ty) \in \mathbb{P}$ for all $x,y \in A$ are trivially fulfilled. Therefore, there exists a unique best proximity point x^* in A, such that for $x_0 \in A$ and $x_{n+1} = Tx_n$ the sequence $\{x_{2n}\}$ converges to x^*. □

We will finish with an example where one cannot use Theorem 8. However, we can apply Theorem 7.

Example 3. Let us consider the Banach space $(\mathbb{R}^2, \|\cdot\|_2)$. Let A and B be

$$A = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq x\},$$
$$B = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, -x-1 \leq y \leq -1\},$$

and let $T : A \cup B \to A \cup B$ be

$$T(x,y) = \begin{cases} (x, -1-y^2), & (x,y) \in A \text{ and } 0 \leq x < 1, \\ (x,-1), & (x,y) \in A \text{ and } x = 1, \\ (x,(y+1)^2), & (x,y) \in B \text{ and } 0 \leq x < 1, \\ (x,0), & (x,y) \in B \text{ and } x = 1. \end{cases}$$

Let us define $\mathbb{P} = \{((x_1,y_1),(x_2,y_2)) \in A \times B : x_1 = x_2\}$. Then, by Theorem 7, there exists a best proximity point.

Indeed, X is a complete metric space, $A, B \subset X$ and $A \cap B = \emptyset$. Due to A and B being convex sets, it follows that the pair (A,B) has the UC property [20]. It is clear that T is a cyclic map and that $\text{dist}(A,B) = 1$.

Due to $x_1 = x_2 = x$ for any $((x_1,y_1),(x_2,y_2)) \in \mathbb{P}$ and the fact that T preserves the value of the x coordinate, we can see that conditions (i), (ii) and (iii) are fulfilled. We will show that (iv) is also fulfilled.

For $((x,y_1),(x,y_2)) \in \mathbb{P}$, we express y_1 and y_2 as

$$\begin{array}{ll} y_1 = ux, & u \in [0,1], \\ y_2 = -1 - vx, & v \in [0,1]. \end{array}$$

Then by

$$\begin{aligned} \|T(x,y_1) - T(x,y_2)\|_2 &= \sqrt{(-1 - y_1^2 - (y_2+1)^2)^2} = |(y_2+1)^2 + 1 + y_1^2| = (y_2+1)^2 + 1 + y_1^2 \\ &= v^2x^2 + 1 + u^2x^2 = v^2x^2 + u^2x^2 + x + 1 - x \le vx^2 + ux^2 + x + 1 - x \\ &= x|ux - (-1 - vx)| + 1 - x = x|y_1 - y_2| + 1 - x \\ &= x\|(x,y_1) - (x,y_2)\|_2 + (1-x)\text{dist}(A,B), \end{aligned}$$

we see that

$$\|T(x,y_1) - T(x,y_2)\|_2 \le x\|(x,y_1) - (x,y_2)\|_2 + (1-x)\text{dist}(A,B).$$

If we take $k = \frac{1}{2}$ in condition (iv), then we can use

$$n(x,y) = \begin{cases} \min\left\{2n+1 : n \in \mathbb{N} \cup \{0\}, 2n+1 \ge \log_x\left(\frac{1}{2}\right)\right\}, & 0 \le x < 1, \\ 1, & x = 1 \end{cases}$$

in order to have

$$\|T^{n(x,y_1)}(x,y_1) - T^{n(x,y_1)}(x,y_2)\|_2 \le \frac{1}{2}\|(x,y_1) - (x,y_2)\|_2 + \left(1 - \frac{1}{2}\right)\text{dist}(A,B).$$

Therefore, condition (iv) is fulfilled. Then, by Theorem 7, we conclude that there exists a best proximity point $(x,y) \in \mathbb{P}$, such that by initiating the iteration process with an arbitrary $(x_0,y_0) \in A$, we get that $(x_{2n},y_{2n}) \to (x,y)$ and $T^2(x,y) = (x,y)$. Clearly, $(x,y) = (x_0,0)$.

Furthermore, by (a) we can conclude that for any $(x_1,y_1) \in A$ such that $x_1 = x_0$, have that $(T^{2n}x_1, T^{2n}y_1) \to (x_0,0)$. This quickly follows due to $((x_1,y_1),(Tx_0,Ty_0))$ and $((x_0,y_0),(Tx_1,Ty_1))$ being both in $\mathbb{P}$.

However, for $x_1 \ne x_2$ we have that neither $((x_1,y_1),(Tx_2,Ty_2))$ nor $((x_2,y_2),(Tx_1,Ty_1))$ are elements of $\mathbb{P}$ and there does not exist a $z \in X$ that fulfills the condition in (c). Therefore, we cannot claim that the best proximity point is unique. And indeed, every point of the form $(x,0), x \in [0,1]$ is a best proximity point.

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