ON THE SPECTRAL STABILITY OF CNOIDAL AND SNOIDAL PERIODIC WAVES OF THE NONLINEAR WAVE EQUATION *

STANIMIR ST. STANEV

ABSTRACT: *We study the spectral stability of periodic traveling wave solutions of the cnoidal and snoidal type for the nonlinear wave equation. First we need to obtain the required spectral information about the operator of linearization. Then we investigate the index of stability and evaluate some quantities.*

KEYWORDS: *periodic traveling waves, linear stability, nonlinear wave equation*

DOI: https://doi.org/10.46687/BLES4089

1 Introduction

In the present paper we study the following nonlinear wave equation

$$u_{tt} - u_{xx} - g(u) = 0, \qquad g \in C^3(\mathbb{R}). \tag{1}$$

For equation (1), stability of traveling wave solutions on the whole line, was considered in [10].

Recently, the linear stability of traveling wave solutions for the second order in time nonlinear differential equations has been studied extensively [1, 7, 11, 12] . In [11] the question of the stability analysis for the second order in time PDEs is reduced to the study of stability of quadratic pencils in the form $\lambda^2 + 2c\lambda\partial_x + \mathscr{H} = 0$, where $\mathscr{H}$ is a self-adjoint operator. If $\mathscr{H}$ has a simple negative eigenvalue and a simple eigenvalue at zero, the authors in [11] derived the index of stability and the abstract results were applied to Boussinesq equation, Klein-Gordon equation and beam equation.

In this paper we are interested in the stability of periodic traveling wave solutions of (1) with respect to perturbations that are periodic and of the same period as the corresponding wave solutions. First we need to obtain the required spectral information about the operator of linearization. Then we investigate the index of stability defined in [8].

The paper is organized as follows. In Section 2, we prove the existence of periodic traveling waves. In Section 3, we set up the linearized problem and give the general abstract result that we use. In Section 4, we consider the stability of periodic traveling waves of cnoidal type. In the last section we consider the stability of periodic traveling waves of snoidal type.

2 Periodic traveling waves

We look for periodic traveling wave solutions for equation (1) in the form $u(t,x) = \phi(x+ct)$ where $c \neq \pm 1$. It is assumed that ϕ is smooth in $\mathbb{R}$. Replacing in (1), we get

$$w\phi'' + g(\phi) = 0, \tag{2}$$

with $w = 1 - c^2$. Integrating (2) yields

$$\frac{\phi'^2}{2} + \frac{G(\phi)}{w} = a \tag{3}$$

where $a \in \mathbb{R}$ is a constant of integration and $G(\phi) = \int_0^\phi g(s)ds$.

*Partially supported by Scientific Research Grant RD-08-103/2024 of Shumen University.

3 Linear stability overview

Take $u = \phi(x+ct)+v(t,x+ct))$, where $v(t,x)$ is a periodic function with respect to x with a fundamental period T, and plug it into (1). Using (2) and ignoring all terms in the form $O(v^2)$, we get the following linear equation for v

(4) $$v_{tt} + 2cv_{tx} + Hv = 0,$$

where

$$H = -w\partial_x^2 - g'(\phi).$$

If we consider the eigenvalue problem associated with (4)), that is $v = e^{\lambda t}V$, we arrive at

$$\lambda^2 V + 2c\lambda V_x + HV = 0.$$

Definition 1. *We say that the traveling wave solution ϕ is linearly unstable, if there exist a T-periodic function $\psi \in D(H)$ and $\lambda : \Re\lambda > 0$, such that*

(5) $$\lambda^2 V + 2c\lambda V_x + HV = 0.$$

Otherwise, we say that ϕ is stable.

We can write an equivalent to (5) Hamiltonian eigenvalue problem, namely

(6) $$\mathscr{J}\mathscr{H}\vec{V} = \lambda\vec{V}, \ \ \vec{V} = \begin{pmatrix} u \\ v \end{pmatrix} \in X \times X,$$

where

$$\mathscr{J} = \begin{pmatrix} 0 & 1 \\ -1 & -2c\partial_x \end{pmatrix}, \ \ \mathscr{H} = \begin{pmatrix} H & 0 \\ 0 & 1 \end{pmatrix}.$$

We use the instability index count theory, as developed in [8]. We present a corollary, which is enough for our purposes. For eigenvalue problem in the form (7), we assume that $\mathscr{H} = \mathscr{H}^*$ has $dim(Ker(\mathscr{H}) < \infty$, and also a finite number of negative eigenvalues, $n(\mathscr{H})$, a quantity sometimes referred to as Morse index of the operator $\mathscr{H}$. In addition, $\mathscr{J}^* = -\mathscr{J}$ We consider the eigenvalue problem

(7) $$\mathscr{J}\mathscr{H}\vec{\mathbf{U}} = \lambda\vec{\mathbf{U}}.$$

Let k_r be the number of positive eigenvalues of the spectral problem (7) (i.e. the number of real instabilities or real modes), k_c be the number of quadruplets of eigenvalues with non-zero real and imaginary parts, and k_i^-, the number of pairs of purely imaginary eigenvalues with negative Krein-signature. For a simple pair of imaginary eigenvalues $\pm i\mu, \mu \neq 0$, and the corresponding eigenvector $\vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, the Krein index is $sgn(\langle \mathscr{H}\vec{z}, \vec{z} \rangle)$, see [2], p. 267.

Also of importance in this theory is a finite dimensional matrix $\mathscr{D}$, which is obtained from the adjoint eigenvectors for (7). More specifically, consider the generalized kernel of $\mathscr{J}\mathscr{L}$

$$gKer(\mathscr{J}\mathscr{H}) = span[(Ker(\mathscr{J}\mathscr{H}))^l, l = 1, 2, \ldots].$$

Assume that $dim(gKer(\mathcal{J}\mathcal{H})) < \infty$ (note that under minimal Fredholm assumptions on $\mathcal{J}, \mathcal{H}$, this is indeed the case). Select an orthonormal basis in $gKer(\mathcal{J}\mathcal{H}) \ominus Ker(\mathcal{J}\mathcal{H}) = span[\eta_j, j = 1,\dots,N]$. Then $\mathcal{D} \in M_{N\times N}$ is defined via

$$\mathcal{D} := \{\mathcal{D}_{ij}\}_{i,j=1}^{N} : \mathcal{D}_{ij} = \langle \mathcal{L}\eta_i, \eta_j\rangle.$$

Then, following [8], we have the following formula, relating the number of "instabilities" or Hamiltonian index of the eigenvalue problem (7) and the Morse indices of the operators $\mathcal{L}$ and $\mathcal{D}$

$$k_{Ham} := k_r + 2k_c + 2k_i^- = n(\mathcal{L}) - n(\mathcal{D}). \tag{8}$$

It is well-known that the first five eigenvalues of $\Lambda = -\partial_y^2 + 6k^2 sn^2(y,k)$, with periodic boundary conditions on $[0, 4K(k)]$ are simple. These eigenvalues and corresponding eigenfunctions are:

$$\begin{aligned}
&\nu_0 = 2 + 2k^2 - 2\sqrt{1-k^2+k^4}, && \phi_0(y) = 1 - (1 + k^2 - \sqrt{1-k^2+k^4})sn^2(y,k),\\
&\nu_1 = 1 + k^2, && \phi_1(y) = cn(y,k)dn(y,k) = sn'(y,k),\\
&\nu_2 = 1 + 4k^2, && \phi_2(y) = sn(y,k)dn(y,k) = -cn'(y,k),\\
&\nu_3 = 4 + k^2, && \phi_3(y) = sn(y,k)cn(y,k) = -k^{-2}dn'(y,k),\\
&\nu_4 = 2 + 2k^2 + 2\sqrt{1-k^2+k^4}, && \phi_4(y) = 1 - (1 + k^2 + \sqrt{1-k^2+k^4})sn^2(y,k).
\end{aligned}$$

4 Stability of cnoidal waves

Consider the case $g(s) = -s + s^3$. We have the following ordinary differential equation for φ,

$$-w\varphi'' + \varphi - \varphi^3 = 0, \tag{9}$$

where $w = 1 - c^2$. Integrating once, we get

$$\varphi'^2 = \frac{1}{2w}(-\varphi^4 + 2\varphi^2 + a), \tag{10}$$

where a is a constant of integration. Then, for $w > 0$ and $a > 0$ up to a translation, we obtain the respective explicit formulas

$$\varphi(x) = \varphi_0 cn(\alpha x, \kappa), \tag{11}$$

where

$$\kappa^2 = \frac{\varphi_0^2}{2\varphi_0^2 - 2}, \quad \alpha^2 = \frac{2\varphi_0^2 - 2}{2w} = \frac{1}{w(2\kappa^2 - 1)}. \tag{12}$$

For the operator $H = -w\partial_x^2 + 1 - 3\varphi^2$ using that $sn^2(y) + cn^2(y) = 1$, we get

$$\begin{aligned}
H &= -w\partial_x^2 + 1 - 3\varphi_0^2 cn^2(\alpha x, \kappa)\\
&= w\alpha^2\left[-\partial_y^2 + 6\kappa^2 sn^2(y,\kappa) - (1 + 4\kappa^2)\right] = w\alpha^2[\Lambda_1 - (1 + 4\kappa^2)],
\end{aligned}$$

where $y = \alpha x$.

It follows that the first three eigenvalues of the operator H, equipped with periodic boundary condition on $[0,4K(k)]$ are simple and zero is the third eigenvalue.

Hence $n(\mathscr{H})=2$, kernel of $\mathscr{H}$ is one dimensional and spanned by $(\varphi',0)$.

We have $\xi=\begin{pmatrix}\varphi'\\0\end{pmatrix}\in\ker\mathscr{H}$. We now proceed to find the generalized kernel of $\mathscr{J}\mathscr{H}$, i.e. the adjoint eigenvectors. Recall that we are only interested in those outside $Ker(\mathscr{H})$. We looking for adjoint, $\mathscr{J}\mathscr{H}\begin{pmatrix}f\\g\end{pmatrix}=\begin{pmatrix}\varphi'\\0\end{pmatrix}$, which is equivalent to

$$\begin{cases} g=\varphi' \\ -Hf-2cg'=0. \end{cases}$$

Hence, we get $Hf=-2c\varphi''$ and $f=-2cH^{-1}\varphi''$ and $\eta=\begin{pmatrix}f\\g\end{pmatrix}$. We need to look further at a second order adjoints, that is solutions of $\mathscr{J}\mathscr{H}\eta_1=\eta$. A necessary condition for the solvability of this last problem is $\begin{pmatrix}-2c\partial_x & -1\\1 & 0\end{pmatrix}\eta\perp\xi$, which is equivalent to $4c^2\langle H^{-1}\varphi'',\varphi''\rangle+\langle\varphi',\varphi'\rangle=0$.

We have $\mathscr{H}\begin{pmatrix}f\\g\end{pmatrix}=\begin{pmatrix}-2c\varphi''\\\varphi'\end{pmatrix}$. Thus

$$\langle\mathscr{H}\begin{pmatrix}f\\g\end{pmatrix},\begin{pmatrix}f\\g\end{pmatrix}\rangle=4c^2\langle H^{-1}\varphi'',\varphi''\rangle+\langle\varphi',\varphi'\rangle.$$

Now we will estimate $\langle H^{-1}\varphi'',\varphi''\rangle$ and $\langle\varphi',\varphi'\rangle$. We have the following representation

$$\langle H^{-1}\varphi'',\varphi''\rangle=\frac{1}{w^2}\langle H^{-1}\varphi,\varphi\rangle+\frac{1}{2w^2}\langle\varphi,\varphi\rangle-\frac{1}{2w}\langle\varphi',\varphi'\rangle.$$

First, we will compute $\langle H^{-1}\varphi,\varphi\rangle$. We have $H\varphi'=0$. The function

$$\psi(x)=\varphi'(x)\int_0^x\frac{1}{\varphi'^2(s)}ds,\quad\begin{vmatrix}\varphi' & \psi\\\varphi'' & \psi'\end{vmatrix}=1$$

is also solution of $H\psi=0$. Formally, since φ' has zeros using the identity

$$\frac{1}{sn^2(y,\kappa)}=-\frac{1}{dn(y,\kappa)}\frac{\partial}{\partial_y}\frac{cn(x,\kappa)}{sn(y,\kappa)}$$

and integrating by parts, we get

$$\psi(x)=\frac{1}{\alpha^2\varphi_0}\left[cn(\alpha x)-\alpha\kappa^2sn(\alpha x,\kappa)dn(\alpha x,\kappa)\int_0^x\frac{1+cn^2(\alpha s,\kappa)}{dn^2(\alpha s,\kappa)}ds\right].$$

After integrating by parts, we get

$$\langle H^{-1}\varphi,\varphi\rangle=-\frac{1}{w}\langle\varphi^3,\psi\rangle+\frac{\varphi^2(T)+\varphi(0)^2}{2w}\langle\varphi,\psi\rangle+C_\varphi\langle\varphi,\psi\rangle. \tag{13}$$

Similarly as in [3], integrating by parts yields

$$\langle\psi'',\varphi\rangle=2\psi'(T)\varphi(T)+\langle\psi,\varphi''\rangle.$$

Using that $H\varphi = -2\varphi^3$, we get

$$\langle \psi, \varphi^3 \rangle = -w\psi'(T)\varphi(T).$$

We have

$$C_\varphi = -\frac{\varphi''(T)}{2w\psi'(T)}\langle \varphi, \psi \rangle + \frac{\varphi^2(T) - \varphi^2(0)}{2w}.$$

With this, we get

$$\langle H^{-1}\varphi, \varphi \rangle = \psi'(T)\varphi(T) + \frac{\varphi^2(T)}{w}\langle \varphi, \psi \rangle - \frac{\varphi''(T)}{2w\psi'(T)}\langle \varphi, \psi \rangle^2$$

and

$$\langle H^{-1}\varphi, \varphi \rangle = -\frac{2}{\alpha}\frac{E^2(\kappa) - 2(1-\kappa^2)E(\kappa)K(\kappa) + (1-\kappa^2)K^2(\kappa)}{(2\kappa^2-1)E(\kappa) + (1-\kappa^2)K(\kappa)}.$$

By direct estimates, we have

$$\begin{cases} ||\varphi||^2 = \frac{\varphi_0^2}{\alpha}\frac{4[E(\kappa)-(1-\kappa^2)K(\kappa)]}{\kappa^2} \\ ||\varphi'||^2 = 4\alpha\varphi_0^2\frac{(2\kappa^2-1)E(\kappa)+(1-\kappa^2)K(\kappa)}{3\kappa^2}. \end{cases}$$

Finally, we get

$$\begin{aligned} ||\varphi'||^2 + 4c^2\langle H^{-1}\varphi'', \varphi'' \rangle &= \frac{8}{\alpha w^2}\Big[\frac{(2\kappa^2-1)E(\kappa)+(1-\kappa^2)K(\kappa)}{3(2\kappa^2-1)^2} \\ &\quad - \frac{\kappa^2(1-\kappa^2)K^2(\kappa)}{(2\kappa^2-1)^2[(2\kappa^2-1)E(\kappa)+(1-\kappa^2)K(\kappa)]}c^2\Big] \end{aligned}$$

If the above expression is negative, then the right side of (8) is odd number. With this we proved the following theorem

Theorem 1. *Periodic traveling wave solutions of cnoidal type are spectrally unstable for all*

$$c^2 > \frac{[(2\kappa^2-1)E(\kappa) + (1-\kappa^2)K(\kappa)]^2}{3\kappa^2(1-\kappa^2)K^2(\kappa)}.$$

5 Stability of snoidal waves

In this case $g(s) = s - s^3$.

We have the following ordinary differential equation for φ,

(14) $$-w\varphi'' - \varphi + \varphi^3 = 0.$$

Integrating once, we get

(15) $$\varphi'^2 = \frac{1}{2w}\varphi^4 - \frac{1}{w}\varphi^2 + 2a.$$

The solution of (14) is given bay

(16) $$\varphi(x) = \varphi_0 sn(\alpha x, \kappa),$$

where α and κ are parameters to be determined.

Since the fundamental period of elliptic function sn is $4K(\kappa)$, then the fundamental period of $\varphi(x)$ is

$$2T = \frac{4K(\kappa)}{\alpha}.$$

We have

$$\begin{aligned} H &= -w\partial_x^2 - 1 + 3\varphi_0^2 sn^2(\alpha x, \kappa) \\ &= w\alpha^2 \left[-\partial_y^2 + 6\kappa^2 sn^2(y, \kappa) - (1+\kappa^2)\right], \end{aligned}$$

where $y = \alpha x$.

It follows that the zero is the second eigenvalue H, equipped with periodic boundary condition on $[0, 4K(k)]$ are simple and zero is the second eigenvalue.

We have $\xi = \begin{pmatrix} \varphi' \\ 0 \end{pmatrix} \in \ker \mathscr{H}$. We now proceed to find the generalized kernel of $\mathscr{J}\mathscr{H}$, i.e. the adjoint eigenvectors. Recall that we are only interested in those outside $Ker(\mathscr{H})$. We looking for adjoint, $\mathscr{J}\mathscr{H}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \varphi' \\ 0 \end{pmatrix}$, which is equivalent to

$$\begin{cases} g = \varphi' \\ -Hf - 2cg' = 0. \end{cases}$$

Hence, we get $Hf = -2c\varphi''$ and $f = -2cH^{-1}\varphi''$ and $\eta = \begin{pmatrix} f \\ g \end{pmatrix}$. We need to look further at a second order adjoints, that is solutions of $\mathscr{J}\mathscr{H}\eta_1 = \eta$. A necessary condition for the solvability of this last problem is $\begin{pmatrix} -2c\partial_x & -1 \\ 1 & 0 \end{pmatrix}\eta \perp \xi$, which is equivalent to $4c^2\langle H^{-1}\varphi'', \varphi''\rangle + \langle \varphi', \varphi' \rangle = 0$.

We have $\mathscr{H}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} -2c\varphi'' \\ \varphi' \end{pmatrix}$. Thus

$$\langle \mathscr{H}\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}\rangle = 4c^2\langle H^{-1}\varphi'', \varphi''\rangle + \langle \varphi', \varphi' \rangle.$$

Since $\langle \varphi'', \phi_0(\alpha x)\rangle = 0$, then $\langle H^{-1}\varphi'', \varphi''\rangle \geq 0$ and

$$n(\mathscr{D}) = n(||\varphi'||^2 + 4c^2\langle H^{-1}\varphi'', \varphi''\rangle) = 0.$$

Hence the right side of (8) is odd number

$$n(\mathscr{H}) - n(\mathscr{D}) = 1 - 0 = 1.$$

REFERENCES:

[1] J. Bronski, M. Johnson, T. Kapitula, An instability index theory for quadratic pencils and applications, *Comm. Math. Phys.*, **327(2)** (2014), 521-550.

[2] T. Kapitula, P. Kevrekidis, B. Sandstede, Counting eigenvalues via the Krein signature in infinite-dimensional Hamiltonian systems, Physica D, 195(2004), 263-282

[3] B. Deconinck, T. Kapitula, *On the spectral and orbital stability of spatially periodic stationary solutions of generalized Korteweg–de Vries equations,* Hamiltonian partial differential equations and applications, 285-322, Fields Inst. Commun. 75, 2015.

[4] M. Grillakis, *Analysis of the linearization around a critical point of an infinite-dimensional Hamiltonian system*, Comm. Pure Appl. Math. **43** (1990), 299–333.

[5] M. Grillakis, J. Shatah, W. Strauss, *Stability of solitary waves in the presence of symmetry I*, J. Funct. Anal. **74** (1987), 160–197.

[6] M. Grillakis, J. Shatah, W. Strauss, *Stability of solitary waves in the presence of symmetry II*, J. Funct. Anal. **94** (1990), 308–348.

[7] S. Hakkaev, M. Stanislavova, A. Stefanov, Linear stability analysis for periodic traveling waves of the Boussinesq equation and the KGZ system, *Proc. Roy. Soc. Edinburgh Sect. A*, **144(03)** (2014), 455-489.

[8] Z. Lin, C. Zeng, *Instability, index theorem, and exponential trichotomy for linear Hamiltonian PDEs*, available at https://arxiv.org/abs/1703.04016, to appear as *Mem. Amer. Math. Soc.*.

[9] F. Natali, *Unstable snoidal waves*, J. London Math. Soc.

[10] M. J. Plaza, J. Stubbe, L. Vazquez, *Existence and stability of travelling waves in (1+1) dimensions*, J. Phys. A: Math. Gen. **23** (1990), 695-705.

[11] M. Stanislavova, A. Stefanov, Linear stability analysis for traveling waves of second order in time PDEs, *Nonlinearity*, **25** (2012), 2625-2654.

[12] M. Stanislavova, A. Stefanov, Spectral stability analysis for special solutions of second order in time PDEs: the higher dimensional case, *Physica D*, **262** (2013), 1-13.

[13] M. Weinstein, *Existence and dynamic stability of solitary-wave solutions of equations arising in long wave propagation*, *Commun. Partial Diff. Eqns.* **12** (1987), 11–33.