APPLICATIONS OF THE NONSELFAJOINT OPERATOR THEORY AND THE SOLITON THEORY*

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ABSTRACT: *In this paper we consider some applications of the connection between the two mathematical theories – the nonselfadjoint operator theory and the soliton theory, for obtaining new solutions of nonlinear differential equations.*

KEYWORDS: *Nonselfadjoint operator, dissipative operator, operator colligation, triangular model, open system, coupling, multiplicative integral, solitonic combination, nonlinear differential equation*

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1 Introduction

This paper is dedicated to the description of an unified approach for solving of different nonlinear differential equations (the Korteweg-de Vries equation, the Schrödinger equation, the Heisenberg equation, the Sine-Gordon equation, the Davey-Stewartson equation) presented in the paper [1], using the connection between the soliton theory and the commuting nonselfadjoint operator theory. The connection between these two mathematical theories is established by M.S. Livšic and Y.Avishai in their paper [8]. In [1] G.S. Borisova has presented an approach to the inverse scattering problem and to the wave equations, based on the Livšic operator colligation theory (or vessel theory) in the case of commuting bounded nonselfadjoint operators in a Hilbert space, when one of the operators belongs to a larger class of nondissipative operators with asymptotics of the corresponding nondissipative curves. With the help of this approach the generalized Gelfand-Levitan-Marchenko equation of the cases of different differential equations (mentoned above) are derived, relations between the wave equations of the input and the output of the generalized open systems, corresponding to some nonlinear differential equations are obtained. We derive what kind of differential equations are satisfied by the components of the input and the output of the corresponding generalized open systems. It turns out that the components of the input and the output satisfy the Sturm-Liouville differential equation from the form

$$Ly = -y'' + q(x)y = \lambda y$$

in the case of the nonlinear Schrödinger equation and the 3-dimensional differential equation from the form

$$Ly = \frac{d^3y}{dy^3} - p(x)\frac{dy}{dx} - q(x)y = \lambda y$$

in the case of the Korteweg-de Vries equation.

The Livšic operator colligation theory investigates nonselfadjoint operators, based on the colligation theory. This implies that instead of the nonselfadjoint operators M.S. Livšic considers the introduced so-called operator colligation (for one operator or several operators), which can be

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considered as a generalization of the nonselfadjoint operator or several operators. To the introduced operator colligation there corresponds the so-called characteristic operator function and the corresponding open system, which has been investigated.

Let us consider the case of two commuting bounded nonselfadjoint operators A, B in a Hilbert space H with finite dimensional imaginary parts, The subspace $G = G_A + G_B$ (where $G_A = (A - A^*)H$, $G_B = (B - B^*)H$) is the non-Hermitian subspace of the pair (A,B), G_A, G_B are the non-Hermitian subspaces of A and B correspondingly. Let the operators A and B be embedded in a regular colligation (an every pair (A,B) of commuting nonselfadjoint operators on H with finite-dimensional imaginary parts can be always embedded in a commutative regular colligation (see, for example, [4]))

$$X = (A,B;H,\widetilde{\Phi},E;\sigma_A,\sigma_B,\gamma,\widetilde{\gamma}) \tag{1.1}$$

where σ_A, σ_B, γ, $\widetilde{\gamma}$ are bounded linear selfadjoint operators in another Hilbert space E , $\widetilde{\Phi} : H \longrightarrow E$ is bounded linear operator, satisfying the conditions

$$(A - A^*)/i = \widetilde{\Phi}^*\sigma_A\widetilde{\Phi}, \quad (B - B^*)/i = \widetilde{\Phi}^*\sigma_B\widetilde{\Phi}, \tag{1.2}$$

$$\sigma_A\Phi B^* - \sigma_B\Phi A^* = \gamma\Phi, \tag{1.3}$$

$$\sigma_A\widetilde{\Phi}B - \sigma_B\widetilde{\Phi}A = \widetilde{\gamma}\widetilde{\Phi}, \tag{1.4}$$

$$\widetilde{\gamma} - \gamma = i(\sigma_A\widetilde{\Phi}\widetilde{\Phi}^*\sigma_B - \sigma_B\widetilde{\Phi}\widetilde{\Phi}^*\sigma_A). \tag{1.5}$$

If $\Phi H = E$ and $\ker\sigma_A \cap \ker\sigma_B = \{0\}$, the colligation is called a strict colligation. A colligation is said to be commutative if $AB = BA$. Strict commutative colligations are regular (see [6, 7]). Here we consider the case of $\dim E < +\infty$.

To a given commutative regular colligation (1.1) there corresponds the following generalized open system (introduced in [4]) from the form

$$\begin{cases} i\frac{1}{\varepsilon}\frac{\partial}{\partial t}f + Af = \widetilde{\Phi}^*\sigma_A u, \\ i\frac{1}{\delta}\frac{\partial}{\partial x}f + Bf = \widetilde{\Phi}^*\sigma_B u, \\ v = u - i\widetilde{\Phi}f, \end{cases} \tag{1.6}$$

where ε, δ are complex constant, the vector functions $u = u(x,t)$, $v = v(x,t)$ with values in E and $f = f(x,t)$ with values in H are the collective input, the collective output, and the collective state correspondingly.

For the commutative regular colligation X the equations (1.6) of open system are compatible if and only if the input $u = u(x,t)$ and the output $v = v(x,t)$ satisfy the following partial differential equations correspondingly (see Theorem 3.3, [4])

$$\sigma_B\left(-i\frac{1}{\varepsilon}\frac{\partial u}{\partial t}\right) - \sigma_A\left(-i\frac{1}{\delta}\frac{\partial u}{\partial x}\right) + \gamma u = 0, \tag{1.7}$$

$$\sigma_B\left(-i\frac{1}{\varepsilon}\frac{\partial v}{\partial t}\right) - \sigma_A\left(-i\frac{1}{\delta}\frac{\partial v}{\partial x}\right) + \widetilde{\gamma}v = 0. \tag{1.8}$$

The equations (1.7), (1.8) are the matrix wave equations. To the generalized open system (1.6) there correspond the more general collective motions of the form

$$T(x,t) = e^{i(\varepsilon tA+\delta xB)}, \quad T^{*-1}(x,t) = e^{i(\overline{\varepsilon}tA^*+\overline{\delta}xB^*)}. \tag{1.9}$$

It is evident that the vector function (or so-called open field, following the terminology of M.S. Livšic) $f(x,t) = T(x,t)h$ $(h \in H)$ satisfies the system (1.6) with identically zero input and an arbitrary initial state $f(0,0) = h$ $(h \in H)$.

The preliminary results, concerning the application of the connection between the soliton theory and the commuting nonselfadjoint operator theory, are obtained in [4] in the case when one of the operators belongs to the the larger class of nonselfadjoint nondissipative operators–couplings of dissipative and antidissipative operators with absolutely continuous real spectra (introduced and investigated by G.S. Borisova, K.P. kirchev in [2, 5]). These preliminary results allow to expand the idea for solutions of the KdV equation (obtained by M.S. Livšic and Y. Avishai in [8] for the dissipative operator B with zero limit $\lim\limits_{x\to+\infty}(e^{ixB}f, e^{ixB}f) = 0$ $(f \in H)$) in the case of the considered larger class of nondissipative operators.

Let the operators A and B be commuting linear bounded nonselfadjoint operators in a separable Hilbert space H. Let us suppose that A and B satisfy the conditions:

(I) the operators A and B have finite-dimensional imaginary parts (i.e. the so-called non-hermitian subspaces $G_A = (A-A^*)H$ and $G_B = (B-B^*)H$ of the operators A and B are finite-dimensional subspaces);

(II) the operator B is a coupling of dissipative and antidissipative operators with absolutely continuous real spectra (and consequently, there exists $\lim\limits_{x\to+\infty}(e^{ixB}h, e^{ixB}h) \neq 0$ $(h \in H)$, [5]).

Without loss of generality, we can assume that the operator B is the triangular model

$$\begin{aligned} Bf(w) = \alpha(w)f(w) - i\int_{a'}^{w} f(\xi)\Pi(\xi)S^*\Pi^*(w)d\xi + \\ +i\int_{w}^{b'} f(\xi)\Pi(\xi)S\Pi^*(w)d\xi + i\int_{a'}^{w} f(\xi)\Pi(\xi)L\Pi^*(w)d\xi, \end{aligned} \tag{1.10}$$

where $a' = -l$, $b' = l$, i.e. $\Delta = [-l,l]$, $H = \mathbf{L}^2(\Delta;\mathbb{C}^p)$, $f = (f_1, f_2, \dots, f_p) \in H = \mathbf{L}^2(\Delta;\mathbb{C}^p)$, $L: \mathbb{C}^m \longrightarrow \mathbb{C}^m$, $\det L \neq 0$, $L^* = L$, $L = J_1 - J_2 + S + S^*$,

$$J_1 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, J_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix}, S = \begin{pmatrix} 0 & 0 \\ \widehat{S} & 0 \end{pmatrix}, \tag{1.11}$$

r is the number of positive eigenvalues and $m-r$ is the number of negative eigenvalues of the matrix L, $\Pi(w)$ is a measurable $p \times m$ $(1 \leq p \leq m)$ matrix function on Δ, whose rows are linearly independent at each point of a set of positive measure, the matrix function $\widetilde{\Pi}(w) = \Pi^*(w)\Pi(w)$ satisfies the conditions $tr\,\widetilde{\Pi}(w) = 1$, $\widetilde{\Pi}(w)J_1 = J_1\widetilde{\Pi}(w)$, $||\widetilde{\Pi}(w_1) - \widetilde{\Pi}(w_2)|| \leq C|w_1 - w_2|^{\alpha_1}$ for all $w_1, w_2 \in \Delta$ for some constant $C > 0$, α_1 is an appropriate constant with $0 < \alpha_1 \leq 1$ (see [5]), (where $||\ ||$ is the norm in $\mathbb{C}^m$) and the function $\alpha: \Delta \longrightarrow \mathbb{R}$ satisfies the conditions:

(i) the function $\alpha(w)$ is continuous strictly increasing on Δ;

(ii) the inverse function $\sigma(u)$ of $\alpha(w)$ is absolutely continuous on $[a,b]$ $(a = \alpha(a'),\ b = \alpha(b'))$;

(iii) $\sigma'(u)$ is continuous and satisfies the relation $|\sigma'(u_1) - \sigma'(u_2)| \leq C|u_1 - u_2|^{\alpha_2}$, $(0 < \alpha_2 \leq 1)$ for all u_1, $u_2 \in [a,b]$ and for some constant $C > 0$.

The imaginary part of the operator B from (1.10) satisfies the condition $(B-B^*)/i=\Phi^*L\Phi$, where the operator $\Phi : H \longrightarrow H$ is defined by the equality

$$\Phi f(w)=\int_{a'}^{b'} f(w)\Pi(w)dw. \tag{1.12}$$

The existence of the wave operators $W_\pm(B^*,B)=s-\lim_{x\to\pm\infty} e^{ixB^*}e^{-ixB}$ of the couple of operators (B,B^*) as strong limits has been established and their explicit form has been obtained in [5] and [3], i.e.

$$W_\pm(B^*,B)=s-\lim_{x\to\pm\infty} e^{ixB^*}e^{-ixB}=\widetilde{S}^*_\mp\widetilde{S}_\mp. \tag{1.13}$$

The explicit form of the operators $\widetilde{S}_\mp$ on the right hand side of the relation (1.13) for the operator B with triangular model (1.10) has been obtained in [5] in the terms of the multiplicative integrals and the finite dimensional analogue of the classical gamma function (introduced in [5]).

In [4] (Theorem 2.1) it has been obtained that if a bounded linear operator $\rho:\mathbf{L}^2(\Delta;\mathbb{C}^p)\longrightarrow \mathbf{L}^2(\Delta;\mathbb{C}^p)$ commutes with the operator of multiplication with an independent variable in the space $\mathbf{L}^2(\Delta;\mathbb{C}^p)$, then the operator M, defined in $\mathbf{L}^2(\Delta;\mathbb{C}^p)$ by the equality

$$M=\int_0^\infty e^{-ixB^*}\rho\frac{B-B^*}{i}e^{ixB}dx \tag{1.14}$$

(as a strong limit), satisfies the relation

$$B^*M-MB=\rho(B^*-B) \tag{1.15}$$

For the existence of the integral in (1.14) and the equality (1.15) we essentially use the existence and the explicit form of the limit $s-\lim_{x\to+\infty} e^{-ixB^*}e^{ixB}=\widetilde{S}^*_+\widetilde{S}_+$, which follows from (1.13) and has been obtained in [5].

Finally it has to be mentioned that for different appropriate choices of the complex constants ε, δ and an appropriate relation between the operator A and the operator B of a coupling from the form (1.10) tne applications for the nonlinear differential equations, mentined above, based on the connection between the opertor theory and the soliton theory, are obtained in [4, 1].

2 The special case of the input and the output of the open system, corresponding to the Schödinger equation

In this section we consider appropriate operators A, B and the corresponding generalized open system with appropriate choice of the complex constants ε, δ, generating solutions of the nonlinear Schrödinger equation in the special case of separated variables in the input, the internal state and the output. We derive what kind of nonlinear differential equations are satisfied by the components of the input and the output of the corresponding open system which are $2m$ vector functions. The obtained equations are as the equations in [10].

Let the operators A, B be commuting linear bounded nonselfadjoint nondissipative operators in a separable Hilbert space H,satisfying the conditions **(I)**, **(II)**. Let the operator B be a coupling of dissipative and antidissipative operators with real absolutely continuous spectra. Without loss of generality we can assume that the coupling B is the triangular model (1.10) when $\Delta = [-l, l]$, $H = \mathbf{L}^2(\Delta, \mathbb{C}^p)$. Let the operators $\Pi(w)$, $Q(w)$, $\widetilde{\Pi}(w)$, L, Φ be stated as in Section 1.

Let the operators $A = bB^2$, B be embedded in the commutative regular colligation X from (1.1), i.e.

$$X = (A = bB^2, B; H = \mathbf{L}^2(\Delta; \mathbb{C}^p), \widetilde{\Phi}, E = \mathbb{C}^{2m}; \sigma_A, \sigma_B, \gamma, \widetilde{\gamma}) \tag{2.1}$$

where σ_A, σ_B, γ, $\widetilde{\gamma}$ are stated as in Section 1. Let us consider the generalized open system, corresponding to X (***in the case when*** $\varepsilon = \mathbf{i}$, $\delta = \mathbf{1}$) from the form (1.6)

$$\begin{cases} \frac{\partial f}{\partial t} + Af = \widetilde{\Phi}^* \sigma_A u \\ i\frac{\partial f}{\partial x} + Bf = \widetilde{\Phi}^* \sigma_B u \\ v = u - i\widetilde{\Phi} f. \end{cases} \tag{2.2}$$

Then the collective motions (1.9) have the form

$$T(x,t) = e^{i(itA + xB)}, \quad T^{*-1}(x,t) = e^{i(-itA^* + xB^*)}. \tag{2.3}$$

Now Theorem 3.3 in [4] shows that collective motions are compatible if and only if the input and the output satisfy (when $\varepsilon = i$, $\delta = 1$) the following partial differential equations (or so called matrix wave equations) correspondingly

$$-\sigma_B \frac{\partial u}{\partial t} + i\sigma_A \frac{\partial u}{\partial x} + \gamma u = 0, \tag{2.4}$$

$$-\sigma_B \frac{\partial v}{\partial t} + i\sigma_A \frac{\partial v}{\partial x} + \widetilde{\gamma} v = 0. \tag{2.5}$$

Let us consider now the special case of separated variables in the input, the output and the state when

$$u(x,t) = e^{-\lambda t} u_\lambda(x), \quad v(x,t) = e^{-\lambda t} v_\lambda(x), \quad f(x,t) = e^{-\lambda t} f_\lambda(x). \tag{2.6}$$

Then the corresponding open system takes the form

$$\begin{cases} -\lambda f_\lambda(x) + A f_\lambda(x) = \widetilde{\Phi} \sigma_A u_\lambda(x) \\ i\frac{df_\lambda(x)}{dx} + B f_\lambda(x) = \widetilde{\Phi} \sigma_B u_\lambda(x) \\ v_\lambda(x) = u_\lambda(x) - i\widetilde{\Phi} f_\lambda(x). \end{cases} \tag{2.7}$$

Consequently, $f_\lambda(x) = (A - \lambda I)^{-1} \widetilde{\Phi} \sigma_A u_\lambda(x)$ and

$$v_\lambda(x) = u_\lambda(x) - i\widetilde{\Phi}(A - \lambda I)^{-1} \widetilde{\Phi} \sigma_A u_\lambda(x) = W_A(\lambda) u_\lambda(x), \tag{2.8}$$

where $W_A(\lambda)$ is the characteristic operator function of the operator A (λ does not belong to the spectrum of the operator A, i.e. λ is a regular point of the characteristic operator function of A). Then direct calculations (using the form and the conditions for the operators σ_A, σ_B, γ, $\widetilde{\Phi}$, described in Section 1) show that the matrix $\widetilde{\gamma}$ has the form

$$\widetilde{\gamma} = \begin{pmatrix} -ibL(\pi_{12} - \pi_{12}^*)L & -ibL\pi_{11}L \\ ibL\pi_{11}L & bL \end{pmatrix}, \tag{2.9}$$

where we have used the block representation of the selfadjoint matrix

$$\widetilde{\Phi}\widetilde{\Phi}^* h = h \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{12}^* & \pi_{22} \end{pmatrix}, \tag{2.10}$$

where the matrices π_{11}, π_{22} satisfy $\pi_{11}^* = \pi_{11}$, $\pi_{22}^* = \pi_{22}$ and they do not depend on x.

Using the denotations

$$u_\lambda(x) = (u_1(x), u_2(x)), \quad v_\lambda(x) = (v_1(x), v_2(x)), \tag{2.11}$$

the compatibility conditions (2.4) and (2.5) for $u_\lambda(x)$ and $v_\lambda(x)$ and straightforward calculations we obtain the next theorem.

Theorem 2.1. *The input $u(x,t) = e^{-\lambda t} u_\lambda(x) = (e^{-\lambda t} u_1(x), e^{-\lambda t} u_2(x))$ and the output $v(x,t) = e^{-\lambda t} v_\lambda(x) = (e^{-\lambda t} v_1(x), e^{-\lambda t} v_2(x))$ of the open system (2.7) satisfy the compatibility conditions*

$$\sigma_A \frac{du_\lambda}{dx} - i(\lambda\sigma_B + \gamma)u_\lambda = 0$$
$$\sigma_A \frac{dv_\lambda}{dx} - i(\lambda\sigma_B + \widetilde{\gamma})v_\lambda = 0$$

and the characteristic operator function $W_A(\lambda) = I - i\widetilde{\Phi}(A - \lambda I)^{-1}\widetilde{\Phi}\sigma_A$ of the operator $A = bB^2$ maps the input $u(x,t)$, satisfying the equations

$$-\frac{d^2u_1}{dx^2} = \frac{\lambda}{b}u_1$$
$$-\frac{d^2u_2}{dx^2} = \frac{\lambda}{b}u_2$$

to the output $v(x,t) = W_A(\lambda)u(x,t)$ which components $v_1(x)$, $v_2(x)$ are solutions of the equations

$$-\frac{d^2v_1}{dx^2} + v_1((L\pi_{11})^2 + iL(\pi_{12} - \pi_{12}^*)) = \frac{\lambda}{b}v_1$$
$$-\frac{d^2v_2}{dx^2} + v_2((L\pi_{11})^2 + iL(\pi_{12} - \pi_{12}^*)) = \frac{\lambda}{b}v_2$$

(when the operator $\widetilde{\Phi}$ satisfies the condition $\pi_{11}L(\pi_{12} - \pi_{12}^*) = (\pi_{12} - \pi_{12}^*)L\pi_{11}$, and $v_\lambda(x) = W_A(\lambda)u_\lambda(x)$).

3 The case when the operator A and B depend on the spatial variable x and the Schödinger equation

The results, obtained in the previous Section 2, can be expanded in the case when the operators A and B depend on the spatial variable x, i.e. for the special case of separated variables in the input, the state, and the output of the generalized open system, connected with obtaining solutions of the nonlinear Schrödinger equation when the operators A and B depend on the spatial variable x and they are prezented in [1]. As in Section 2 there are derived what kind of differential equations are satisfied by the components of the input and the output of the corresponding generalized open system–Sturm-Liouville differential equations with matrix function potentials, which depend on the matrix function $\widetilde{\Phi}(x)\widetilde{\Phi}^*(x)$ (defined bellow).

Let us consider now regular colligations (or vessels) which depend on the spatial variable x. This is the case when the operators A, B, $\widetilde{\Phi}$ depend on the spatial variable x, i.e. the matrix function $\Pi(w)$ depends also on the variable x, i.e. $\Pi = \Pi(w,x)$.

Using the results of M.S. Livšic in the article [9] (Section 3.5) it follows that the commuting operators $A(x)$, $B(x)$ are embedded in the strict colligation

$$X = (A(x), B(x); H, \widetilde{\Phi}(x), E = \mathbb{C}^{2m}; \sigma_A, \sigma_B, \psi(x), \widetilde{\psi}(x)),$$

(where the operator functions $\widetilde{\Phi}(x)$, $\psi(x)$, $\widetilde{\psi}(x)$ are differentiable). Let us consider a generalized open system with the form (analogous to the case when operators A, B do not depend on the variables x and t)

$$(3.1) \qquad \begin{cases} i\frac{1}{\varepsilon}\frac{\partial f(x,t)}{\partial t} + A(x)f(x,t) = \widetilde{\Phi}^*(x)\sigma_A u(x,t) \\ i\frac{1}{\delta}\frac{\partial f(x,t)}{\partial x} + B(x)f(x,t) = \widetilde{\Phi}^*(x)\sigma_B u(x,t) \\ v(x,t) = u(x,t) - i\widetilde{\Phi}(x)f(x,t) \end{cases}$$

($t_0 \le t \le t_1$, $x_0 \le x \le x_1$, $\varepsilon, \delta \in \mathbb{C}$). The colligation conditions now have the form

$$(3.2) \qquad (A(x) - A^*(x))/i = \widetilde{\Phi}^*(x)\sigma_A\widetilde{\Phi}(x), \quad (B(x) - B^*(x))/i = \widetilde{\Phi}^*(x)\sigma_B\widetilde{\Phi}(x),$$

$$(3.3) \qquad \frac{1}{i\overline{\delta}}\sigma_A\frac{d\widetilde{\Phi}(x)}{dx} + \sigma_A\widetilde{\Phi}(x)B^*(x) - \sigma_B\widetilde{\Phi}(x)A^*(x) = \psi(x)\widetilde{\Phi}(x),$$

$$(3.4) \qquad \frac{1}{i\delta}\sigma_A\frac{d\widetilde{\Phi}(x)}{dx} + \sigma_A\widetilde{\Phi}(x)B(x) - \sigma_B\widetilde{\Phi}(x)A(x) = \widetilde{\psi}(x)\widetilde{\Phi}(x),$$

$$(3.5) \qquad \widetilde{\psi}(x) = \psi(x) + i(\sigma_A\widetilde{\Phi}(x)\widetilde{\Phi}^*(x)\sigma_B - \sigma_B\widetilde{\Phi}(x)\widetilde{\Phi}^*(x)\sigma_A).$$

(M.S. Livšic has considered in [9] (Section 3.5) an open system in the case when $\varepsilon = \delta = 1$).

Now the results of M.S. Livšic in [9] imply that

$$(3.6) \qquad \frac{i}{\delta}\frac{dA(x)}{dx}f = A(x)B(x)f - B(x)A(x)f.$$

In [1] the next theorem, concerning the matrix wave equations in this case, is proved.

Theorem 3.1. *The input $u(x,t)$ and the output $v(x,t)$ of the open system (3.1) with $A(x)B(x) = B(x)A(x)$ satisfy the strong compatibility conditions (or matrix wave equations)*

$$(3.7) \qquad \sigma_B\left(-i\frac{1}{\varepsilon}\frac{\partial u}{\partial t}(x,t)\right) - \sigma_A\left(-i\frac{1}{\delta}\frac{\partial u}{\partial x}(x,t)\right) + \psi(x)u(x,t) = 0,$$

$$(3.8) \qquad \sigma_B\left(-i\frac{1}{\varepsilon}\frac{\partial v}{\partial t}(x,t)\right) - \sigma_A\left(-i\frac{1}{\delta}\frac{\partial v}{\partial x}(x,t)\right) + \widetilde{\psi}(x)v(x,t) = 0.$$

Let us consider the operator $B(x)$, presented as a coupling of dissipative and antidissipative operators with real absolutely continuous spectra. Without loss of generality we can assume that the operator $B(x)$ is the triangular model (1.10) in the Hilbert space $H = \mathbf{L}^2(\Delta;\mathbb{C}^p)$ where $\Delta = [-l,l]$. Let the operators $\Pi(w,x)$, $Q(w,x)$, $\widetilde{\Pi}(w,x)$, L, $\Phi(x)$ be like in Section 2, but depending on the other variable x ($x_0 \leq x \leq x_1$). Let the operator $B(x)$ satisfy the condition $B^* = -UBU^*$ ($U : H \longrightarrow H$, $U^*U = UU^* = I$).

Let the operators $A(x) = bB^2(x)$ ($b \in \mathbb{R}$) and $B(x)$ be embedded in the colligation

$$X = (A(x) = bB^2(x), B(x); H = \mathbf{L}^2(\Delta;\mathbb{C}^p), \widetilde{\Phi}, E = \mathbb{C}^{2m}; \sigma_A, \sigma_B, \psi(x), \widetilde{\psi}(x)),$$

where σ_A, σ_B, $\widetilde{\Phi}$ ($\widetilde{\Phi} = \widetilde{\Phi}(x)$) are defined by the equalities

$$\sigma_A = \begin{pmatrix} 0 & bL \\ bL & 0 \end{pmatrix}, \quad \sigma_B = \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 0 \\ 0 & bL \end{pmatrix}, \quad \widetilde{\Phi} = (\Phi \quad \Phi B^*).$$

Let us consider the case of generalized open system (3.1) when $\varepsilon = i$, $\delta = 1$

$$\text{(3.9)} \qquad \begin{cases} \frac{\partial f(x,t)}{\partial t} + A(x)f(x,t) = \widetilde{\Phi}^*(x)\sigma_A u(x,t) \\ i\frac{\partial f(x,t)}{\partial x} + B(x)f(x,t) = \widetilde{\Phi}^*(x)\sigma_B u(x,t) \\ v(x,t) = u(x,t) - i\widetilde{\Phi}(x)f(x,t) \end{cases}$$

($t_0 \leq t \leq t_1$, $x_0 \leq x \leq x_1$). Then the corresponding colligation conditions are as (3.2)—(3.5) (with $\delta = 1$).

Next we determine the form of the matrix functions $\psi(x)$ and $\widetilde{\psi}(x)$. i.e. we can consider $\psi(x)$ as a matrix function from the form $\psi(x) = \sigma_1(x) + \gamma$. Let us present $\psi(x)$ in the form

$$\text{(3.10)} \qquad \psi(x) = \begin{pmatrix} bL\psi_{11}(x)L & ibL\psi_{12}(x)L \\ -ibL\psi_{12}^*(x)L & bL \end{pmatrix} = \begin{pmatrix} \widehat{\psi}_{11} & \widehat{\psi}_{12} \\ \widehat{\psi}_{12}^* & bL \end{pmatrix},$$

where $\psi_{11}^* = \psi_{11}$. Now using the block representation of the matrix function $\widetilde{\Phi}(x)\widetilde{\Phi}(x)^*$

$$\text{(3.11)} \qquad \widetilde{\Phi}(x)\widetilde{\Phi}^*(x) = \begin{pmatrix} \pi_{11}(x) & \pi_{12}(x) \\ \pi_{12}^*(x) & \pi_{22}(x) \end{pmatrix}$$

($\pi_{11}^*(x) = \pi_{11}(x)$, $\pi_{22}^*(x) = \pi_{22}(x)$), the form of the matrices σ_A, σ_B we obtain

$$\text{(3.12)} \qquad \begin{aligned} \widetilde{\psi}(x) &= \psi(x) + i\left(-\begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \pi_{11}(x) & \pi_{12}(x) \\ \pi_{12}^*(x) & \pi_{22}(x) \end{pmatrix} \begin{pmatrix} 0 & bL \\ bL & 0 \end{pmatrix} + \right. \\ &\left. + \begin{pmatrix} 0 & bL \\ bL & 0 \end{pmatrix} \begin{pmatrix} \pi_{11}(x) & \pi_{12}(x) \\ \pi_{12}^*(x) & \pi_{22}(x) \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix} \right) = \\ &= \psi - i \begin{pmatrix} bL(\pi_{12} - \pi_{12}^*)L & bL\pi_{11}L \\ -bL\pi_{11}L & 0 \end{pmatrix} = \\ &= \begin{pmatrix} \widehat{\psi}_{11} - ibL(\pi_{12} - \pi_{12}^*)L & \widehat{\psi}_{12} - ibL\pi_{11}L \\ \widehat{\psi}_{12}^* + ibL\pi_{11}L & \widehat{\psi}_{22} \end{pmatrix} = \\ &= \begin{pmatrix} bL\widetilde{\psi}_{11}(x)L & ibL\widetilde{\psi}_{12}(x)L \\ -ibL\widetilde{\psi}_{12}^*(x)L & bL \end{pmatrix}, \end{aligned}$$

where $\widetilde{\psi}_{11}^*(x) = \widetilde{\psi}_{11}(x)$, $\widetilde{\psi}_{12}(x)$ are matrices, which depend on x.

Theorem 3.2. *The input* $u(x,t) = e^{-\lambda t}u_\lambda(x) = e^{-\lambda t}(u_1(x), u_2(x))$ *and the output* $v(x,t) = e^{-\lambda t}v_\lambda(x) = e^{-\lambda t}(v_1(x), v_2(x))$ *of the open system (3.9) satisfy the compatibility conditions*

$$\begin{array}{l} \frac{du_\lambda(x)}{dx} - i\sigma_A^{-1}(\lambda\sigma_B + \psi(x))u_\lambda(x) = 0, \\ \frac{dv_\lambda(x)}{dx} - i\sigma_A^{-1}(\lambda\sigma_B + \widetilde{\psi}(x))v_\lambda(x) = 0 \end{array} \tag{3.13}$$

and the characteristic operator function $W_A(\lambda) = I - i\widetilde{\Phi}(A - \lambda I)^{-1}\widetilde{\Phi}^*\sigma_A$ *maps the input* $u_\lambda(x) = (u_1(x), u_2(x))$*, satisfying the Sturm-Liouville equations*

$$-\frac{d^2u_1}{dx^2} + u_1((L\psi_{12})^2 - L\psi_{11} - L\frac{d\psi_{12}}{dx}) = \frac{\lambda}{b}u_1,$$

$$-\frac{d^2u_2}{dx^2} + u_2((L\psi_{12})^2 - L\psi_{11} + L\frac{d\psi_{12}}{dx}) = \frac{\lambda}{b}u_2$$

to the output $v_\lambda(x) = (v_1(x), v_2(x)) = W_A(\lambda)u_\lambda$*, which are solutions of the following Sturm-Liouville equations*

$$-\frac{d^2v_1}{dx^2} + v_1((L\widetilde{\psi}_{12})^2 - L\widetilde{\psi}_{11} - L\frac{d\widetilde{\psi}_{12}}{dx}) = \frac{\lambda}{b}v_1,$$

$$-\frac{d^2v_2}{dx^2} + v_2((L\widetilde{\psi}_{12})^2 - L\widetilde{\psi}_{11} + L\frac{d\widetilde{\psi}_{12}}{dx}) = \frac{\lambda}{b}v_2$$

in the case when the operator functions $\widetilde{\Phi}(x)$*,* $\psi(x)$*,* $\widetilde{\psi}(x)$ *satisfy the conditions*

$$\psi_{12} = \psi_{12}^*, \quad -\psi_{12}L\psi_{11} + \psi_{11}L\psi_{12}^* + \frac{d\psi_{11}}{dx} = 0,$$

$$\widetilde{\psi}_{12} = \widetilde{\psi}_{12}^*, \quad -\widetilde{\psi}_{12}L\widetilde{\psi}_{11} + \widetilde{\psi}_{11}L\widetilde{\psi}_{12}^* + \frac{d\widetilde{\psi}_{11}}{dx} = 0$$

(for $x \in [x_0, x_1]$*).*

In [10] realizing solutions of the Korteveg-de Vries equation by generating the Korteveg-de Vries vessel (constructing the Sturm-Liouville operator with analytic potentials) is presented, using another approach. This idea has been applied for obtaining solutions of evolutionary nonlinear Schrödinger equation.

Finally, it is worth to mention that the presented approach in this paper can be applied to other nonlinear differential equations, using an appropriate choice of the operators, the complex constants in the generalized open systems, an appropriate dimension of the additional space E in the colligation. For example, in the paper [1] (Section 8, Section 9.) G.S. Borisova has obtained results about the input and the output of the open system and other differential equation – the Korteweg-de Vries equation.

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