FOCAL CURVES OF CLOSED SADDLE CURVES*

CVETELINA L. DINKOVA, RADOSTINA P. ENCHEVA

ABSTRACT: *We consider closed space curves on the hyperbolic paraboloid (saddle), generated by three classes of well known plane curves: the epicycloid, the hypocycloid and the curves that are the orthogonal projections of a toroidal helix. The focal curves of the considered space saddle curves were investigated.*

KEYWORDS: *Hyperbolic paraboloid (Saddle), Saddle curves, Closed curves, Focal curves, Generalized focal curves, Frenet-Seret frame, Curvature, Torsion, Toroidal helix, Epicycloid, Hypocycloid*

DOI: https://doi.org/10.46687/CXQL7965

1 Introduction

In geometry and topology, a closed curve is a connected, continuous curve that starts and ends at the same point. A simple closed curve is a closed curve that doesn't intersect itself. The closed curves have a variety of forms and shapes, and they are essential in many branches of science and mathematics, such as engineering and physics, as well as computer graphics. A circle is among the simplest forms of a simple closed curve. Other closed curves are cardioids, epicycloids, and hypocycloids.

We continue the research from our earlier work [3] in this paper. Firstly, we establish a way to take a given plane curve and use it to create a new space curve that lies on a hyperbolic paraboloid (saddle). We will call those curves saddle curves. The saddle surfaces and the curves laying on them have many applications in architecture design, roof construction, and daily life. Next, we apply this method to the closed planar curves mentioned above. Furthermore, we also examine the orthogonal projection of a known space curve called a toroidal helix onto the Euclidean plane. The focal curves of the generated saddle space curve, which lies on a hyperbolic paraboloid, are then constructed. In the end, we obtain a generalized focal curve of a closed plane curve, which is completely distinct from its evolute (a focal curve). Differential-geometric invariants such as Euclidean curvatures, shape curvatures, and focal curvatures play an essential role in the curve examination. In the text that follows, we find some relations between the differential-geometric invariant of the considered curves.

2 Preliminaries

The basic concepts of the classical differential geometry of curves and surfaces in two and three dimensional Euclidean space are introduced in this section. More details can be found in the books "Modern Differential Geometry of Curves and Surfaces" (see [9]) and "Differential Geometry of Curves and Surfaces" (see [4]).

Definition 2.1. *[4, p.18] A parameterized differentiable curve is a differentiable map $\alpha : I \to \mathbb{E}^3$ of an open interval $I \subseteq \mathbb{R}$ of the real line $\mathbb{R}$ into $\mathbb{E}^3$.*

The word "differentiable" in this definition means that α is a correspondence which maps each $t \in I$ into a point $\alpha(t) = (x(t), y(t), z(t)) \in \mathbb{E}^3$, in such a way that the functions $x(t), y(t), z(t)$ are differentiable. The variable t is called a parameter of the curve.

*Partially supported by Scientific Research Grant RD– 08–104/30.01.2024 of Shumen University.

Definition 2.2. *[4, p.22] A parameterized differentiable curve $\alpha : I \to \mathbb{E}^3$ is said to be regular if $\dot{\alpha}(t) = \frac{d\alpha(t)}{dt} \neq 0$ for all $t \in I$.*

We use the symbols $\dot{\alpha} = \dfrac{d\alpha(t)}{dt}$, $\ddot{\alpha} = \dfrac{d\dot{\alpha}(t)}{dt}$ and etc. for a differentiation about an arbitrary parameter t. The scalar product of two vector functions $x(t) = (x_1(t), x_2(t), x_3(t))$ and $y(t) = (y_1(t), y_2(t), y_3(t))$ is given by $\langle x(t), y(t)\rangle = x_1(t)y_1(t) + x_2(t)y_2(t) + x_3(t)y_3(t)$. The norm of the vector function $x(t)$ is given by $\|x(t)\| = \sqrt{\langle x, x\rangle} = \sqrt{x_1^2(t) + x_2^2(t) + x_3^2(t)}$ for $t \in \mathbb{R}$.

For given $t_0 \in I$, the arc-length of a regular parameterized curve $\alpha : I \to \mathbb{E}^3$ from the point t_0 is given by the equality $s(t) = \int_{t_0}^{t} \|\dot{\alpha}(t)\| dt$, where $\|\dot{\alpha}(t)\| = \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}$. Since $\dot{\alpha}(t) \neq 0$, the arc-length function $s = s(t)$ is a differentiable function of t and $ds/dt = \|\dot{\alpha}(t)\|$.

A map $\alpha : I \to \mathbb{E}^3$ is called a curve of class C^k (C^k-curve) if each of the coordinate functions in the expression $\alpha(t) = (x(t), y(t), z(t))$ has continuous derivatives up to order k. We say that α is of class C^0 if it is simply continuous. If the map α is one-to-one, then the curve α is said to be simple. Our investigations are restricted to regular curves with linearly independent derivatives of order from one to n in $\mathbb{E}^n$, $n = 2, 3$. We will call that curves Frenet curves.

The image of any parameterized curve in the Euclidean plane $\mathbb{E}^2 \equiv O\vec{\mathbf{e}}_1\vec{\mathbf{e}}_2$ under an orientation-preserving affine map in $\mathbb{E}^2$ is also a parameterized curve in $\mathbb{E}^2$. Now, we will discuss the differential-geometric invariants of Frenet plane curves with respect to the group of orientation-preserving rigid motions.

A regular parameterized curve $\alpha : [a,b] \to \mathbb{E}^2$ is called a closed plane curve if α and all of its derivatives coincide at a and b, that is $\alpha(a) = \alpha(b)$, $\dot{\alpha}(a) = \dot{\alpha}(b)$, $\ddot{\alpha}(a) = \ddot{\alpha}(b)$, $\dddot{\alpha}(a) = \dddot{\alpha}(b), \ldots$

The curve α is simple if it has no further self-intersections, that is, if $t_1, t_2 \in [a,b], t_1 \neq t_2$, then $\alpha(t_1) \neq \alpha(t_2)$.

Let us consider a Frenet plane curve $\alpha : I \to \mathbf{E}^2$ of class C^3 that is defined on the open interval $I \subseteq \mathbb{R}$ by

$$\alpha(t) = (x(t), y(t), 0). \tag{1}$$

Definition 2.3. *[9, p.3] A **complex structure** J is a linear map $J : \mathbb{E}^2 \to \mathbb{E}^2$ given by $J(p_1, p_2) = (-p_2, p_1)$. Geometrically, J is a rotation by $\pi/2$ in a counterclockwise direction.*

Definition 2.4. *[9, p.11] Let $\alpha : I \to \mathbb{E}^2$ be a Frenet plane curve. The **signed curvature** K of α is given by the formula $K(t) = \dfrac{\langle \ddot{\alpha}(t), J\dot{\alpha}(t)\rangle}{\|\dot{\alpha}(t)\|^3}$. The function $R = \dfrac{1}{K}$ is called a **radius of curvature** of α.*

Remark 2.1. *The signed curvature K defined by the equation in Definition 2.4 above is an invariant under orientation-preserving rigid motions in $\mathbb{E}^2$.*

The Frenet-Seret system of a Frenet space curve γ consists of vectors and scalar invariants:

$$T = \frac{\dot{\gamma}}{\|\dot{\gamma}\|}, N = \frac{(\dot{\gamma} \times \ddot{\gamma}) \times \dot{\gamma}}{\|(\dot{\gamma} \times \ddot{\gamma}) \times \dot{\gamma}\|}, B = \frac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|}, \varkappa = \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3}, \tau = \frac{\dot{\gamma}\ddot{\gamma}\dddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2},$$

where the vector invariants T, N, B are known as a tangent, a principal and a binormal unit vector field of γ, respectively, whereas the scalar invariants $\varkappa$ and τ are called a curvature and a torsion of γ, respectively, or Euclidean curvatures.

Definition 2.5. *[9, p.241] A **focal curve** of a Frenet space curve $\gamma : I \to \mathbb{E}^3$ of class C^3 is the curve given by*

$$\mathbf{C}_\gamma(t) = \gamma(t) + c_1(t)\mathbf{N}(t) + c_2(t)\mathbf{B}(t), \tag{2}$$

where $\mathbf{N}$ is a principal unit normal vector field of γ, $\mathbf{B}$ is a binormal unit vector field of γ. The coefficients $c_1(t)$ and $c_2(t)$ are smooth functions called focal curvatures of γ, and given by

$$c_1(t) = \frac{1}{\varkappa(t)}, \quad c_2(t) = -\frac{\frac{d}{dt}\varkappa(t)}{\| \frac{d\gamma(t)}{dt} \| \varkappa(t)^2 \tau(t)} = \frac{\frac{dc_1(t)}{dt}}{\| \frac{d\gamma(t)}{dt} \| \tau(t)}, \tag{3}$$

where $\varkappa(t)$ and $\tau(t)$ are the Euclidean curvatures of γ.

In other words, the focal curve of a Frenet curve γ in the Euclidean space $\mathbb{E}^3$ consists of the centres of its osculating spheres.

Remark 2.2. *The functions $c_1(t)$ and $c_2(t)$ are well defined because $\varkappa(t)$ and $\tau(t)$ are non-zero functions.*

Definition 2.6. *[9, p.438] Let $S \subset \mathbb{E}^3$ be a surface. Then S is a **generalized cylinder** over a curve $\alpha : I \to \mathbb{E}^3$ if S can be parameterized as $S(u,v) = \alpha(u) + v.\vec{\mathbf{q}}$, where $\vec{\mathbf{q}} \in \mathbb{E}^3$ is a fixed vector.*

We consider a case of a right generalized cylinder over the plane curve α, when its rulings are perpendicular to the generating plane curve. That means, the fixed vector is a unit vector $\vec{\mathbf{e}}_3 = (0,0,1) \parallel Oz$ and the parameter v is replaced by the function $f(v) \in \mathbb{R}$, $f(v) \in C^3$. Then the parametrisation of a **right generalized cylinder** is $S_1(u,v) = \alpha(u) + f(v)\vec{\mathbf{e}}_3$.

The term "saddle" is frequently used to describe a hyperbolic paraboloid for rather obvious reasons. The name stems from the fact that its vertical cross sections are parabolas, while the horizontal cross sections are hyperbolas. The Cartesian equation of this surface is $z = -x^2 + y^2$.

3 Previous results

Theorem 3.1. *[8] Let $\alpha(t) = (x(t), y(t), 0)$, $t \in I \subset \mathbb{R}$ be a Frenet plane curve of class C^3 with a nonzero signed curvature, and let $f(t) \in C^3$ be a real-valued function. Suppose that $\vec{\mathbf{e}}_3$ is the unit vector on Oz-axis and*

$$\gamma(t) = \alpha(t) + f(t)\vec{\mathbf{e}}_3, \quad t \in I$$

is a parameterized space curve. Then, $\gamma(t)$ is a Frenet curve whose curvature and torsion are given by

$$\varkappa = \frac{\sqrt{\langle \ddot{\alpha}, J\dot{\alpha} \rangle^2 + \langle \ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha}, \ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha} \rangle}}{\left(\sqrt{\langle \dot{\alpha}, \dot{\alpha} \rangle + \dot{f}^2}\right)^3} \tag{4}$$

$$\tau = \frac{\dddot{f}\langle \ddot{\alpha}, J\dot{\alpha} \rangle + \ddot{f}\langle -J\dot{\alpha}, \dddot{\alpha} \rangle + \dot{f}\langle J\ddot{\alpha}, \dddot{\alpha} \rangle}{\langle \ddot{\alpha}, J\dot{\alpha} \rangle^2 + \langle \ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha}, \ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha} \rangle} \tag{5}$$

The following statement provides us with relations between the Frenet-Seret frame of γ and the signed curvature K of α as well as the derivatives of the arc-length function of α, parameterized about an arbitrary parameter t.

Theorem 3.2. *[3] Let $\alpha = \alpha(t), t \in I \subseteq \mathbb{R}$ be a regular C^3-plane curve in $\mathbb{E}^2$ and $K \neq 0$ is the Euclidean signed curvature of α. Let $\gamma(t) = \alpha(t) + f(t).\vec{\mathbf{e}}_3, t \in I \subseteq \mathbb{R}, f(t) \in C^2$ be the corresponding cylindrical curve over the right generalized cylinder with a base curve α. If T, N, B are vector invariants of γ, then they can be expressed by the derivatives of α, the scalar function f and the unit vector $\vec{\mathbf{e}}_3$ with the following equations*

$$T = \frac{\dot{s} + \dot{f}\vec{e}_3}{\sqrt{\dot{s}^2 + \dot{f}^2}} \tag{6}$$

$$N = \frac{(\dot{s}^2 + \dot{f}^2)\ddot{\alpha} - \frac{1}{2}\frac{d}{dt}(\dot{s}^2 + \dot{f}^2)\dot{\alpha} + \left(\ddot{f}\dot{s}^2 - \dot{f}\frac{d}{dt}\left(\frac{\dot{s}^2}{2}\right)\right)\vec{e}_3}{\sqrt{\dot{s}^2 + \dot{f}^2}\sqrt{\dot{s}^6 K^2 + \|\ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha}\|^2}} \tag{7}$$

$$B = \frac{-J(\ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha}) + \dot{s}^6 K^2 \vec{e}_3}{\sqrt{\dot{s}^6 K^2 + \|\ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha}\|^2}} \tag{8}$$

where $\dot{s}$ is the derivative of the arc-length function $s = s(t)$ of α with respect to an arbitrary parameter t.

The next statement gives us relations between the focal curvatures of γ and the signed curvature of α, parameterized about an arbitrary parameter t.

Theorem 3.3. *[3] Let $\alpha = \alpha(t), t \in I \subseteq \mathbb{R}$ be a regular C^3-plane curve in $\mathbb{E}^2$ and $K \neq 0$ is the Euclidean signed curvature of α. Let $\gamma(t) = \alpha(t) + f(t).\vec{\mathbf{e}}_3, t \in I \subseteq \mathbb{R}, f(t) \in C^2$ be the corresponding cylindrical curve over the right generalized cylinder with a base curve α. If c_1 and c_2 are the focal curvatures of γ, then*

$$c_1(t) = \frac{\sqrt{\dot{s}^2 + \dot{f}^2}}{\sqrt{\dot{s}^6 K^2 + \|\ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha}\|^2}}, \tag{9}$$

$$c_2(t) = \frac{3(\dot{s}^6 K^2 + \|\ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha}\|^2)\frac{d}{dt}(\dot{s}^2 + \dot{f}^2) - (\dot{s}^2 + \dot{f}^2)\frac{d}{dt}(\dot{s}^6 K^2 + \|\ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha}\|^2)}{2\sqrt{\dot{s}^6 K^2 + \|\ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha}\|^2}\,(\dddot{f}\dot{s}^3 K - \langle J(\ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha}), \dddot{\alpha}\rangle)}, \tag{10}$$

where $\dot{s}$ is the derivative of the arc-length function $s = s(t)$ of α with respect to an arbitrary parameter t.

Theorem 3.4. *[3] Let $\alpha = \alpha(t), t \in I \subseteq \mathbb{R}$ be a regular C^3-plane curve in $\mathbb{E}^2$ and $K \neq 0$ is the Euclidean signed curvature of α. Let $\gamma(t) = \alpha(t) + f(t).\vec{\mathbf{e}}_3, t \in I \subseteq \mathbb{R}, f(t) \in C^3$ be the corresponding cylindrical curve over the right generalized cylinder with base curve α and c_1, c_2 are the focal curvatures of γ defined by equations* (9) *and* (10)*. Then the focal curve of γ has vector-parametric representation $C_\gamma(t) = \beta(t) + \tilde{f}(t).\vec{\mathbf{e}}_3$,*

$$\beta(t) = \alpha(t) + \frac{c_1\left((\dot{s}^2 + \dot{f}^2)\ddot{\alpha} - \frac{1}{2}\frac{d}{dt}(\dot{s}^2 + \dot{f}^2)\dot{\alpha}\right) - c_2\sqrt{\dot{s}^2 + \dot{f}^2}\,J(\ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha})}{\sqrt{\dot{s}^2 + \dot{f}^2}\sqrt{\dot{s}^6 K^2 + \|\ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha}\|^2}}, \tag{11}$$

$$\tilde{f}(t) = f(t) + \frac{c_1\left(\ddot{f}\dot{s}^2 - \dot{f}\frac{d}{dt}\left(\frac{\dot{s}^2}{2}\right)\right) + c_2\sqrt{\dot{s}^2 + \dot{f}^2}\,\dot{s}^3 K}{\sqrt{\dot{s}^2 + \dot{f}^2}\sqrt{\dot{s}^6 K^2 + \|\ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha}\|^2}}, \tag{12}$$

where β is ***the generalized focal curve*** *of α, $\dot{s}$ is the derivative of the arc-length function $s = s(t)$ of α with respect to an arbitrary parameter t.*

4 Main results

Let us consider a hyperbolic paraboloid $S: z = -x^2 + y^2$ with a parametric equation

$$S(u,v) = (u, v, v^2 - u^2), \text{ where } u, v \in U \subseteq \mathbb{R}^2. \tag{13}$$

Theorem 4.1. *Let $\alpha = \alpha(t), t \in I \subseteq \mathbb{R}$ be a regular C^3-plane curve in $\mathbb{E}^2$ and $K \neq 0$ is the Euclidean signed curvature of α. Let $\gamma(t) = (x(t), y(t), -x^2(t) + y^2(t)), t \in I \subseteq \mathbb{R}$ be the corresponding saddle space curve over the hyperbolic paraboloid S. If T, N, B are vector invariants of γ, then they can be expressed by the derivatives of α, α^* and the unit vector $\mathbf{\vec{e}}_3$ via the following equations*

$$T = \frac{\dot{s} + 2\langle\alpha^*, \dot{\alpha}\rangle\vec{e}_3}{\sqrt{\dot{s}^2 + 4\langle\alpha^*, \dot{\alpha}\rangle^2}} \tag{14}$$

$$N = \frac{(\dot{s}^2 + 4\langle\alpha^*, \dot{\alpha}\rangle^2)\ddot{\alpha} - \frac{1}{2}\frac{d}{dt}(\dot{s}^2 + 4\langle\alpha^*, \dot{\alpha}\rangle^2)\dot{\alpha} + \left(\frac{d\langle\alpha^*, \dot{\alpha}\rangle}{dt}\dot{s}^2 - \langle\alpha^*, \dot{\alpha}\rangle\frac{d\dot{s}^2}{dt}\right)\vec{e}_3}{\sqrt{\dot{s}^2 + 4\langle\alpha^*, \dot{\alpha}\rangle^2}\sqrt{\dot{s}^6K^2 + \|A\|^2}} \tag{15}$$

$$B = \frac{-J(A) + \dot{s}^6K^2\vec{e}_3}{\sqrt{\dot{s}^6K^2 + \|A\|^2}} \tag{16}$$

where $A = 2[\langle\dot{\alpha}^*, \dot{\alpha}\rangle + \langle\alpha^*, \ddot{\alpha}\rangle]\dot{\alpha} - 2\langle\alpha^*, \dot{\alpha}\rangle\ddot{\alpha}$, $\dot{s}$ is the derivative of the arc-length function $s = s(t)$ of α with respect to an arbitrary parameter t and α^* is the image of the plane curve α about a symmetry with respect to the ordinate axis Oy.

Proof: The proof immediately follows from Theorem 3.2 and the condition $f(t) = -x^2(t) + y^2(t)$. In more details, the vector function $\gamma(t) = (x(t), y(t), -x^2(t) + y^2(t))$ is of class C^2 when its coordinate functions $x(t), y(t)$ are of class C^2. The fact that the functions $x(t), y(t)$ are continuously differentiable up to order 2 immediately follows from the condition that α is a regular C^3-plane curve in $\mathbb{E}^2$. It is easy to see that if $\alpha^*(t) = (-x(t), y(t))$ is the image of the plane curve $\alpha(t) = (x(t), y(t))$ about a symmetry with respect to the ordinate axis Oy. Then the scalar representation of the function $f(t) = -x^2(t) + y^2(t)$ can be written in the form $f(t) = \langle\alpha^*(t), \alpha(t)\rangle$. Recall that $\langle.,.\rangle$ is a notion of the scalar product of two vector functions. Applying differentiations about an arbitrary parameter t we get to the equalities $\dot{f}(t) = 2\langle\alpha^*, \dot{\alpha}\rangle$, $\ddot{f}(t) = 2(\langle\alpha^*, \ddot{\alpha}\rangle + \langle\dot{\alpha}^*, \dot{\alpha}\rangle)$, and $A = \ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha} = 2[\langle\dot{\alpha}^*, \dot{\alpha}\rangle + \langle\alpha^*, \ddot{\alpha}\rangle]\dot{\alpha} - 2\langle\alpha^*, \dot{\alpha}\rangle\ddot{\alpha}$. When we substitute them into equations (6), (7) and (8) we obtain equations (14), (15) and (16). □

In the next statement the focal curvatures of γ can be expressed by the signed curvature of α, the curve α, parameterized about an arbitrary parameter t, α^* that is the image of the plane curve α about a symmetry with respect to the ordinate axis Oy and their derivatives.

Theorem 4.2. *Let $\alpha = \alpha(t), t \in I \subseteq \mathbb{R}$ be a regular C^3-plane curve in $\mathbb{E}^2$ and $K \neq 0$ is the Euclidean signed curvature of α. Let $\gamma(t) = (x(t), y(t), -x^2(t) + y^2(t)), t \in I \subseteq \mathbb{R}$ be the corresponding saddle space curve over the hyperbolic paraboloid S. If c_1 and c_2 are the focal curvatures of γ, then*

$$c_1(t) = \frac{\sqrt{\dot{s}^2 + 4\langle\alpha^*, \dot{\alpha}\rangle^2}}{\sqrt{\dot{s}^6K^2 + \|A\|^2}}, \tag{17}$$

(18) $$c_2(t) = \frac{3(\dot{s}^6K^2 + \|A\|^2)\frac{d}{dt}(\dot{s}^2 + 4\langle\alpha^*, \dot{\alpha}\rangle^2) - (\dot{s}^2 + 4\langle\alpha^*, \dot{\alpha}\rangle^2)\frac{d}{dt}(\dot{s}^6K^2 + \|A\|^2)}{2\sqrt{\dot{s}^6K^2 + \|A\|^2}\,[(3\langle\dot{\alpha}^*, \ddot{\alpha}\rangle + \langle\alpha^*, \dddot{\alpha}\rangle)\dot{s}^3K - \langle J(A), \ddot{\alpha}\rangle]},$$

where $A = 2[\langle\dot{\alpha}^*, \dot{\alpha}\rangle + \langle\alpha^*, \ddot{\alpha}\rangle]\dot{\alpha} - 2\langle\alpha^*, \dot{\alpha}\rangle\ddot{\alpha}$, $\dot{s}$ is the derivative of the arc-length function $s = s(t)$ of α with respect to an arbitrary parameter t and α^* is the image of the plane curve α about a symmetry with respect to the ordinate axis Oy.

Proof: The vector function $\gamma(t) = (x(t), y(t), -x^2(t) + y^2(t))$ is of class C^3 when its coordinate functions $x(t), y(t)$ are of class C^3, which immediately follows from the condition that α is a regular C^3-plane curve in $\mathbb{E}^2$. Substituting derivatives $\dot{f}(t) = 2\langle\alpha^*, \dot{\alpha}\rangle$, $\ddot{f}(t) = 2(\langle\alpha^*, \ddot{\alpha}\rangle + \langle\dot{\alpha}^*, \dot{\alpha}\rangle)$, $\dddot{f}(t) = 2(\langle\alpha^*, \dddot{\alpha}\rangle + 3\langle\dot{\alpha}^*, \ddot{\alpha}\rangle)$ and $A = \ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha} = 2[\langle\dot{\alpha}^*, \dot{\alpha}\rangle + \langle\alpha^*, \ddot{\alpha}\rangle]\dot{\alpha} - 2\langle\alpha^*, \dot{\alpha}\rangle\ddot{\alpha}$ into the equations (9) and (10) in Theorem 3.3 we get to the equations (17) and (18) whence the proof of the theorem is completed. □

According to the next theorem, the focal curve $C_\gamma(t)$ of a saddle space curve γ can be expressed via the signed curvature of α, the curve α, the image $\alpha^* = S_{Oy}(\alpha)$ of the plane curve α about a symmetry with respect to the ordinate axis Oy and their derivatives.

Theorem 4.3. *Let $\alpha = \alpha(t), t \in I \subseteq \mathbb{R}$ be a regular C^3-plane curve in $\mathbb{E}^2$ and $K \neq 0$ is the Euclidean signed curvature of α. Let $\gamma(t) = (x(t), y(t), -x^2(t) + y^2(t))$, $t \in I \subseteq \mathbb{R}$ be the corresponding saddle space curve over the hyperbolic paraboloid S and c_1, c_2 are the focal curvatures of γ defined by equations* (17) *and* (18)*. Then the focal curve of γ has a vector-parametric representation $C_\gamma(t) = \beta(t) + F(t).\vec{\mathbf{e}}_3$,*

(19) $$\beta(t) = \alpha(t) + \frac{c_1\left((\dot{s}^2 + 4\langle\alpha^*, \dot{\alpha}\rangle^2)\ddot{\alpha} - \frac{1}{2}\frac{d}{dt}(\dot{s}^2 + 4\langle\alpha^*, \dot{\alpha}\rangle^2)\dot{\alpha}\right) - c_2\sqrt{\dot{s}^2 + 4\langle\alpha^*, \dot{\alpha}\rangle^2}\,J(A)}{\sqrt{\dot{s}^2 + 4\langle\alpha^*, \dot{\alpha}\rangle^2}\,\sqrt{\dot{s}^6K^2 + \|A\|^2}},$$

(20) $$F(t) = \langle\alpha^*, \alpha\rangle + \frac{c_1\left(2[\langle\dot{\alpha}^*, \dot{\alpha}\rangle + \langle\alpha^*, \ddot{\alpha}\rangle]\dot{s}^2 - \langle\alpha^*, \dot{\alpha}\rangle\frac{d\dot{s}^2}{dt}\right) + c_2\sqrt{\dot{s}^2 + 4\langle\alpha^*, \dot{\alpha}\rangle^2}\,\dot{s}^3K}{\sqrt{\dot{s}^2 + 4\langle\alpha^*, \dot{\alpha}\rangle^2}\sqrt{\dot{s}^6K^2 + \|A\|^2}},$$

*where β is **the generalized focal curve** of α, $A = 2[\langle\dot{\alpha}^*, \dot{\alpha}\rangle + \langle\alpha^*, \ddot{\alpha}\rangle]\dot{\alpha} - 2\langle\alpha^*, \dot{\alpha}\rangle\ddot{\alpha}$, $\dot{s}$ is the derivative of the arc-length function $s = s(t)$ of α with respect to an arbitrary parameter t.*

Proof: The focal curve of γ has a vector-parametric representation $C_\gamma(t) = \beta(t) + \tilde{f}(t).\vec{\mathbf{e}}_3$, according to Theorem 3.4. The scalar function $\tilde{f}(t)$ is given by equation (12), and **the generalised focal curve** β of α is given by equation (11). Then substituting the derivatives $\dot{f}(t) = 2\langle\alpha^*, \dot{\alpha}\rangle$, $\ddot{f}(t) = 2(\langle\alpha^*, \ddot{\alpha}\rangle + \langle\dot{\alpha}^*, \dot{\alpha}\rangle)$ and $A = \ddot{f}\dot{\alpha} - \dot{f}\ddot{\alpha} = 2[\langle\dot{\alpha}^*, \dot{\alpha}\rangle + \langle\alpha^*, \ddot{\alpha}\rangle]\dot{\alpha} - 2\langle\alpha^*, \dot{\alpha}\rangle\ddot{\alpha}$ into the equations (11) and (12) we get the proof of the theorem. □

5 Examples of closed saddle curves in the Euclidean space $\mathbb{E}^3$

5.1 Saddle helixes

A curve that is the orthogonal projection of a toroidal helix (a helix wrapped into a torus) onto the Euclidean plane has a parametric equation $\alpha(t) = (\cos(t)(a + b\cos(nt)), \sin(t)(a + b\cos(nt)))$, $t \in \mathbb{R}$. The images of **the orthogonal projection of a toroidal helix** α (in orange) are presented on Figure 1 for $a = 1, b = 4, n = 8$; $a = 4, b = 1, n = 8$ and $a = b = n = 3$. Then the correspond-

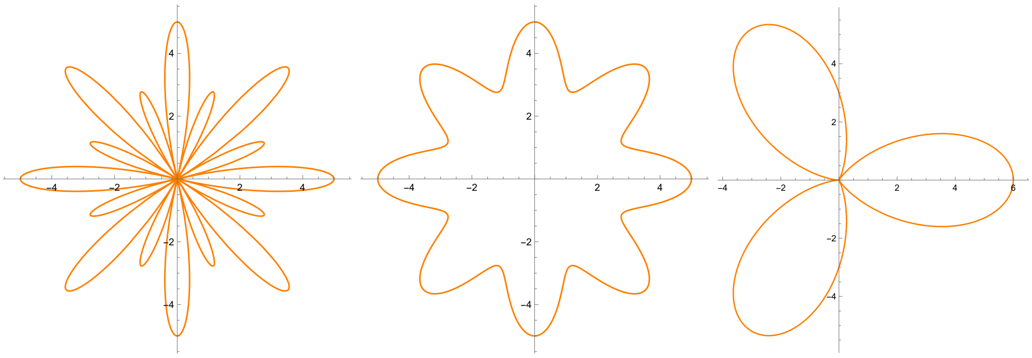

Figure 1

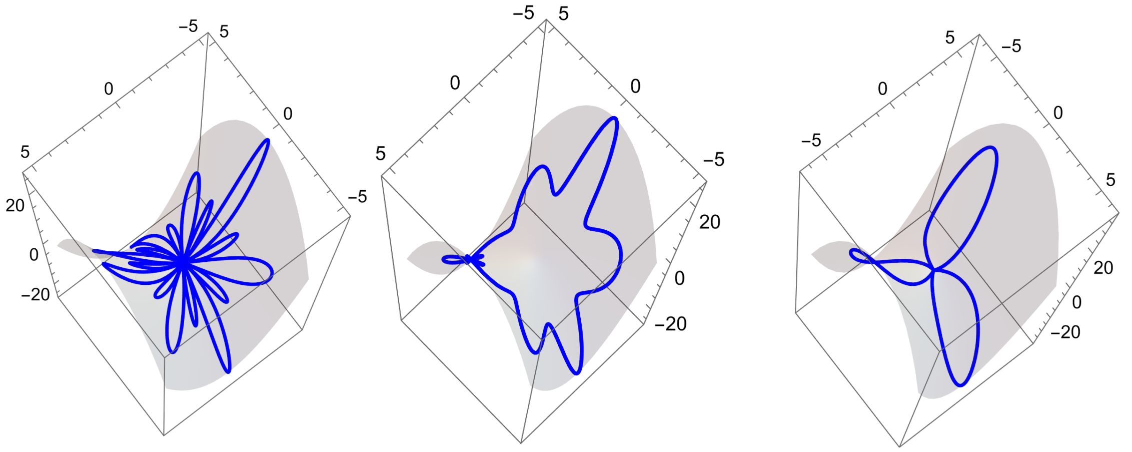

Figure 2

ing space curve on the hyperbolic paraboloid that we will call **a saddle helix** has a parametric representation

$$\gamma(t) = (\cos(t)(a+b\cos(nt)), \sin(t)(a+b\cos(nt)), -\cos(2t)(a+b\cos(nt))^2).$$

The images of **the saddle helixes** γ (in blue) are presented on Figure 2 for $a=1, b=4, n=8$; $a=4, b=1, n=8$ and $a=b=n=3$.

Figure 3 displays the images of **a saddle helix** γ (in blue) over a saddle and its corresponding **focal curve** $\mathbf{C}_\gamma$ (in red) for $a=4, b=1, n=8$.

The image of **the orthogonal projection of a toroidal helix** α (in orange) and its corresponding **generalized focal curve** β (in green) are displayed on Figure 4 for $a=4, b=1, n=8$.

5.2 Saddle epicycloids

The trajectory of a fixed point on a circle, known as an epicycle, which rolls around a given circle without slipping, is a planar curve named an **epicycloid**. The epicycloid and its evolute (focal curve) are similar curves. Now let us examine the epicycloid provided by

$$\alpha(t) = \left((r+R)\cos\left(\frac{rt}{R}\right) - r\cos\left(\frac{t(r+R)}{R}\right), (r+R)\sin\left(\frac{rt}{R}\right) - r\sin\left(\frac{t(r+R)}{R}\right), 0\right),$$

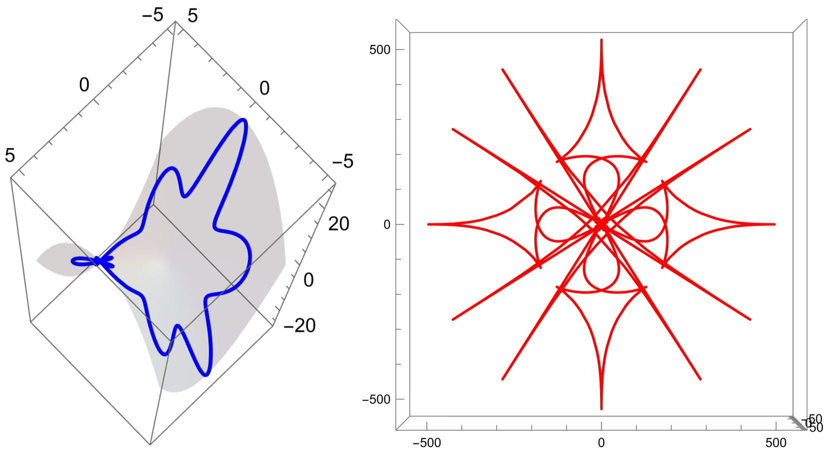

Figure 3

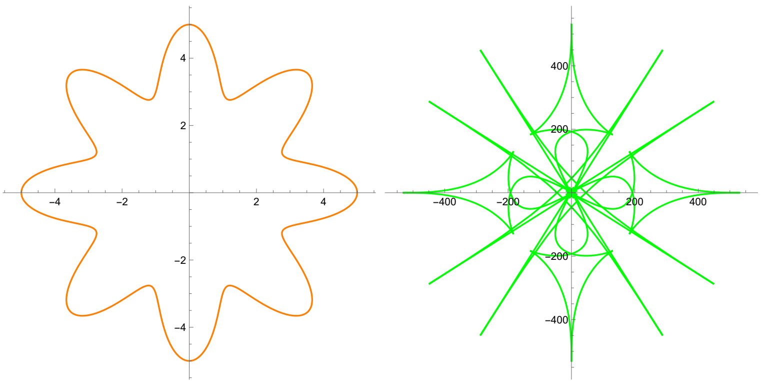

Figure 4

for $t \in [0, 2k\pi]$, $k = 1, 2, \ldots$, where R is the radius of the bigger circle centered at the origin around which the smaller circle with radius r rolls.

The ratio $\frac{r}{R} = m$ determines how the epicycloid looks.

5.2.1 A saddle cardioid and its focal curve

For $R = r$ the epicycloid α is called a **cardioid** and its corresponding curve γ is a **saddle cardioid** with a parametric equation

$$\gamma(t) = \left(-r(\cos(2t) - 2\cos(t)), 2r\sin(t) - r\sin(2t), -r^2(4\cos(2t) - 4\cos(3t) + \cos(4t))\right),$$

for $t \in [0, 2\pi]$. The images of **a saddle cardioid** γ (in blue) over a saddle and its corresponding **focal curve** $\mathbf{C}_\gamma$ (in red) are depicted on Figure 5 for $R = r = 1$. In this case γ is a non-planar curve when $t \neq 0, 2\pi$. The point $\gamma(0) = \gamma(2\pi)$ is a cusp of γ.

The images of a **cardioid** α (in orange) and its corresponding **generalized focal curve** β (in green) are displayed on Figure 6 for $R = r = 1$.

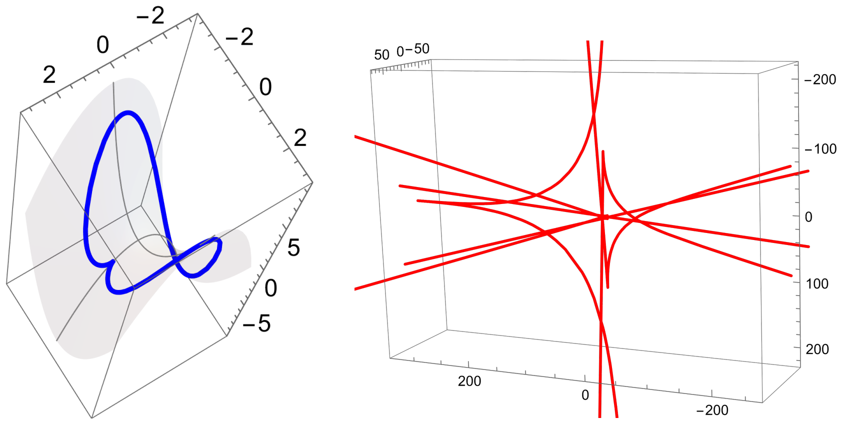

Figure 5

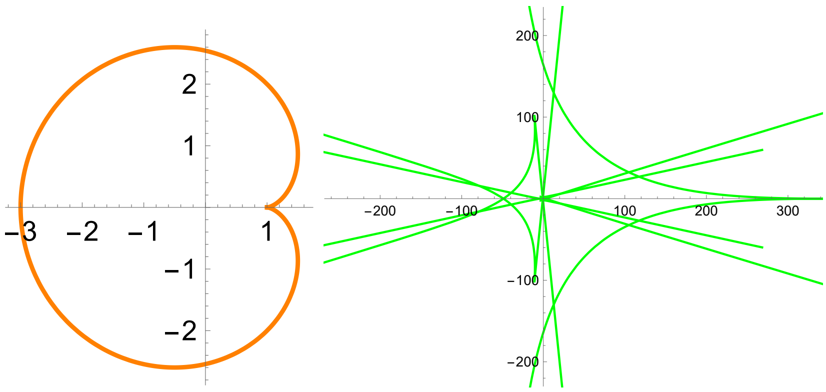

Figure 6

5.2.2 A saddle nephroid and its focal curve

For $R = 2r$ the epicycloid α is called a **nephroid** and its corresponding curve γ is a **saddle nephroid** with a parametric equation

$$(21)\quad \gamma(t) = \left(-r\left(\cos\left(\frac{3t}{2}\right) - 3\cos\left(\frac{t}{2}\right)\right), 4r\sin^3\left(\frac{t}{2}\right), -r^2(9\cos(t) - 6\cos(2t) + \cos(3t))\right)$$

for $t \in [0, 4\pi]$. For $R = 2r$, $r = 1$, Figure 7 shows the pictures of **a saddle nephroid** γ (in blue) over a saddle and its corresponding **focal curve** $\mathbf{C}_\gamma$ (in red).

The images of a **nephroid** α (in orange) and its corresponding **generalized focal curve** β (in green) are depicted on Figure 8 for $R = 2r$, $r = 1$.

5.3 Saddle hypocycloids

A plane curve traced by a fixed point on a circle rolling internally on a given circle is called a **hypocycloid**. The hypocycloid and its evolute (focal curve) are similar curves. Now let us consider the hypocycloid given by

$$\alpha(t) = \left(r\cos\left(\frac{t(R-r)}{R}\right) + (R-r)\cos\left(\frac{rt}{R}\right), (R-r)\sin\left(\frac{rt}{R}\right) - r\sin\left(\frac{t(R-r)}{R}\right), 0\right),$$

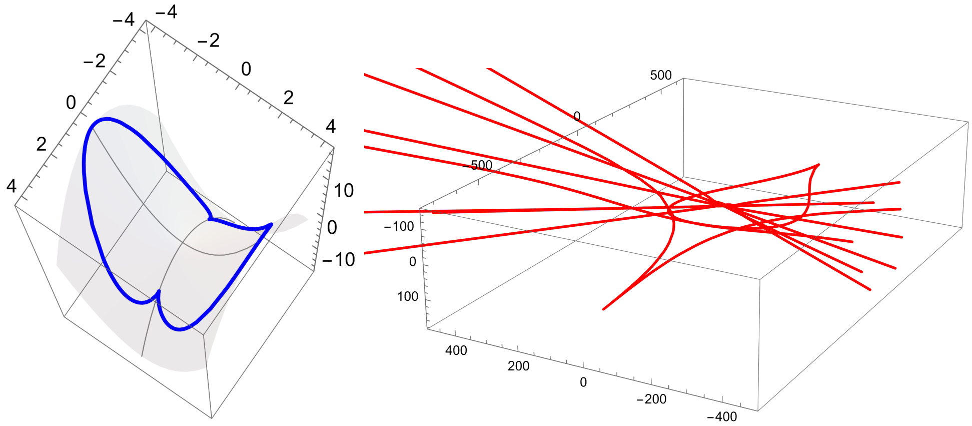

Figure 7

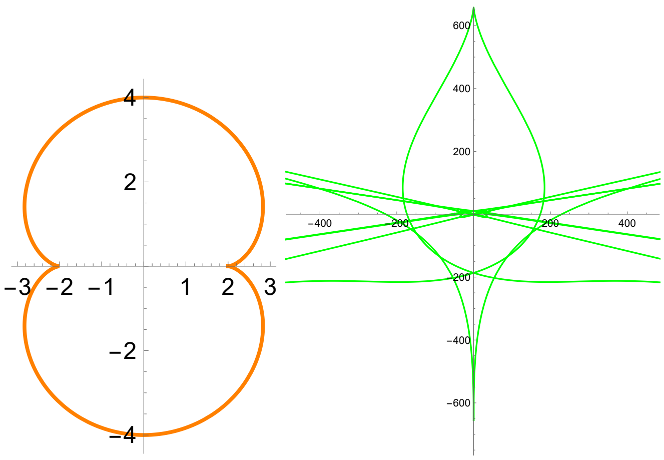

Figure 8

for $t \in [0, 2k\pi]$, $k = 1, 2, \ldots$, where R is the radius of the bigger circle centered at the origin on which the smaller circle with radius r rolls internally.

The shape of the hypocycloids depends on the ratio $\frac{r}{R} = m$.

5.3.1 A saddle deltoid and its focal curve

For $2R = 3r$ the hypocycloid α is called a **deltoid** and its corresponding curve γ is a **saddle deltoid** with a parametric equation

$$\gamma(t) = \left(\frac{1}{2}r\left(2\cos\left(\frac{t}{3}\right) + \cos\left(\frac{2t}{3}\right)\right), -4r\sin^3\left(\frac{t}{6}\right)\cos\left(\frac{t}{6}\right),\right.$$
$$\left.-\frac{1}{4}r^2\left(4\left(\cos\left(\frac{t}{3}\right) + \cos\left(\frac{2t}{3}\right)\right) + \cos\left(\frac{4t}{3}\right)\right)\right), t \in [0, 6\pi]$$

The images of **a saddle deltoid** γ (in blue) over a saddle and its corresponding **focal curve** $\mathbf{C}_\gamma$ (in red) are displayed on Figure 9 for $r = 2$.

Figure 10 shows the images of a **deltoid** α (in orange) and the corresponding **generalized focal curve** β (in green) for $r = 2$.

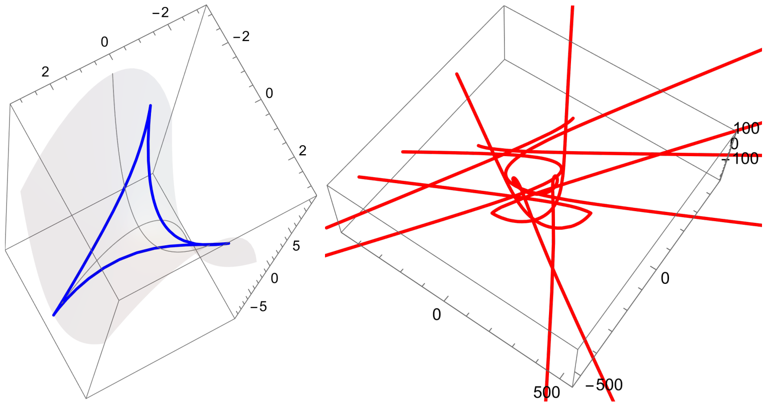

Figure 9

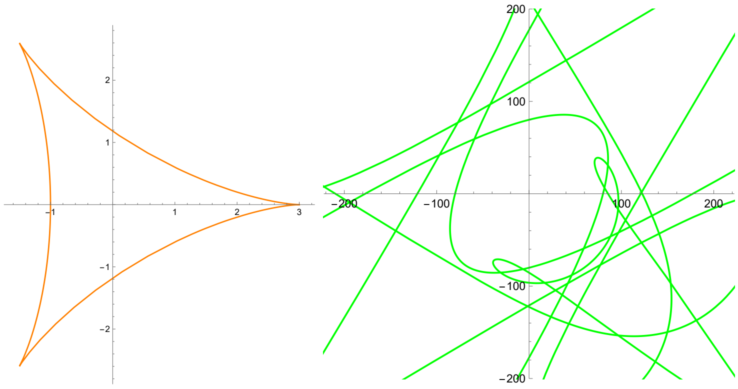

Figure 10

5.3.2 A saddle astroid and its focal curve

For $R = 4r$ the hypocycloid α is called an **astroid** and the corresponding curve γ is **a saddle astroid** with a parametric equation

$$\gamma(t) = \left(4r\cos^3\left(\frac{t}{4}\right), 4r\sin^3\left(\frac{t}{4}\right), -r^2\left(15\cos\left(\frac{t}{2}\right) + \cos\left(\frac{3t}{2}\right)\right)\right), t \in [0, 8\pi]$$

The images of **a saddle astroid** γ (in blue) over a saddle and the corresponding **focal curve** $\mathbf{C}_\gamma$ (in red) are presented on Figure 11 for $r = 1$. Figure 12 displays images of a **astroid** α (in orange) and the corresponding **generalized focal curve** β (in green) for $r = 1$.

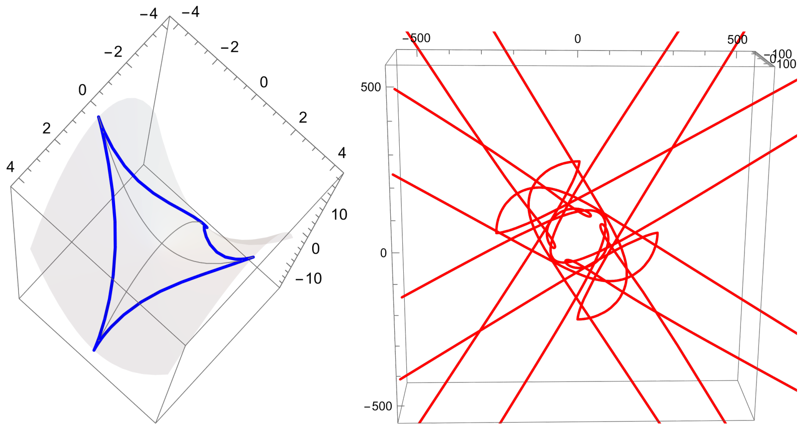

Figure 11

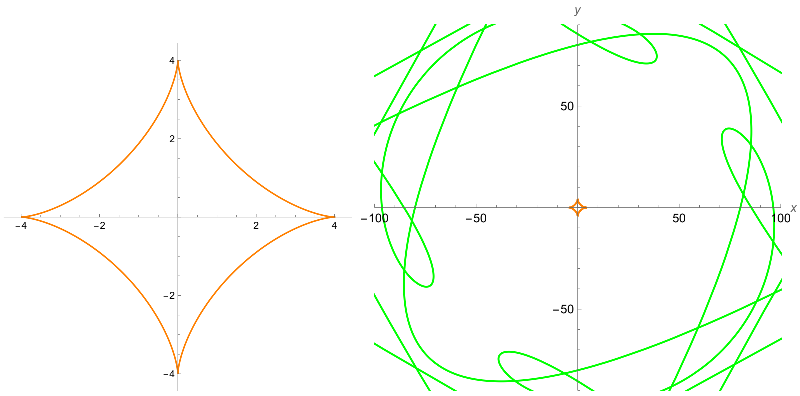

Figure 12

REFERENCES:

[1] Arroyo, J., Garay, O.J., Menca, J.J., When is a peroidic function the curvature of a closed plane curve?, Am. Math. Mon. **115** (5), (2008), 405–414.

[2] Cao, C., Fletcher, A., Ye, Z. (2017). Epicycloids and Blaschke products. Journal of Difference Equations and Applications, 23(9), 1584–1596. https://doi.org/10.1080/10236198.2017.1341499

[3] Dinkova, C. L., Encheva, R. P., and Ali, A. A. (2024). Focal Curves of Closed Toroidal Curves. ArXiv. https://arxiv.org/abs/2410.13476

[4] Do Carmo, M.P., Differential Geometry of Curves and Surfaces: Revised and Updated Second Edition, Dover Publications, INC, (2016)

[5] Encheva, R.P., Georgiev, G.H., Curves on the shape spere, Result. Math. **44** (2003), 279–288.

[6] Encheva, R.P., Georgiev, G.H., Shapes of space curves, J. Geom. Graph. **7** (2003), 145–155.

[7] Encheva, R.P., Georgiev, G.H., Similar Frenet Curves, Result. Math., **55** (3-4) (2009), 359–372. http://www.springerlink.com/content/x6145691n7437r17/

[8] Georgiev, G.H., Encheva, R.P., Dinkova, Cv.L., Geometry of cylindrical curves over plane curves, Applied Mathematical Sciences,**113** (9) (2015), 5637–5649. http://dx.doi.org/10.12988/ams.2015.56456

[9] Gray, A., Abbena, E., Salamon, S., Modern Differential Geometry of Curves and Surfaces, Chapman Hall/CRC, (2006).

[10] Izumiya, S., Takeuchi, N., Special curves and ruled surfaces, Cotributions to Algebra and Geometry, **44** (2003), 203–212.