ON AUGMENTED SPATIO-TEMPORAL DYNAMIC MODE DECOMPOSITION*

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ABSTRACT: *Dynamic mode decomposition (DMD) is a data-driven modeling technique suitable to analyze flow structures in numerical and experimental data. It is widely used to extract temporal information about coherent structures from data. In its standard form, however, it cannot simultaneously extract information on temporal and spatial development, such as wave number and spatial growth rate, which are essential in fully developed flows. Spatio-temporal dynamic mode decomposition (STDMD) is an extension of DMD designed to handle spatio-temporal datasets. The framework is extended to analyze data that varies both spatially and temporally. In this paper we introduce an augmented modification of the STDMD method. It consists of applying delay-embedded coordinates to STDMD sequentially, in time and space. Using this method, dominant frequencies, wavenumbers, and their harmonics can be accurately calculated for complex flows. With different illustrative examples, we demonstrate the applicability of the introduced technique.*

KEYWORDS: *Dynamic mode decomposition, DMD method, spatio-temporal DMD, STDMD, Hankel DMD.* ***MSC****: 65P99, 37M02, 37L65*

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1 Introduction

Dynamic mode decomposition (DMD method), introduced by Schmid in [1], as a method for analyzing data from numerical simulations and laboratory experiments in the fluid dynamics field. It is a mathematical technique for identifying spatiotemporal coherent structures from high-dimensional data. After its introduction, the method is now used in a variety of fields, including video processing [2], epidemiology [3], robotics [4], neuroscience [5], financial trading [6, 7, 8], cavity flows [9, 10] and various jets [11, 12]. For a review of the DMD literature, we refer the reader to [13, 14, 15, 16, 17]. For some recent results on the topics of some derivative DMD techniques, DMD for non-uniformly sampled data, higher order DMD method and parallel implementations of DMD, we recommend to the reader [18, 19, 20, 21, 22, 23, 24, 25, 26], see also [27, 28, 29, 30, 31, 32, 33].

Although the DMD method has been established as a leading technique for extracting temporal information about coherent structures from high-dimensional data, in its standard form it cannot simultaneously extract temporal and spatial evolution information, including wavenumber and spatial growth rate, which is essential in fully developed flows. There is an extension of DMD, the so-called spatio-temporal dynamic mode decomposition (STDMD method), used to handle spatio-temporal datasets, see [34, 35]. STDMD extends the framework for analyzing data that varies both spatially and temporally. In this way, spatial structures can be extracted along with their temporal evolution. A comprehensive mathematical framework for sequential and parallel STDMD approaches is presented in our recent publication [36]. A modification of the STDMD with delay-embedded coordinates is also presented there. The extension is called *parallel delay-embedding DMD*. It simultaneously decomposes spatio-temporal data across both spatial and temporal dimensions, providing insights into the interplay between spatial and temporal dynamics.

In this article, we provide an extension to the STDMD using the delay-embedding approach, but unlike [36] here we use the sequential approach. In contrast to the parallel approach, sequential involves decomposing spatio-temporal data sequentially along the temporal axis, capturing both

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spatial and temporal dynamics separately. It enables the identification of spatial structures evolving over time and their corresponding temporal dynamics. The extension is labeled *augmented sequential STDMD*.

The outline of the paper is as follows: In the rest of Section 1, for completeness of the exposition, we briefly describe DMD, STDMD and augmented DMD frameworks. In Section 2, we propose and discuss the new approach *augmented sequential STDMD*. Numerical results are in Section 3 and the conclusion is in Section 4.

1.1 *Dynamic mode decomposition (DMD)*

In this paragraph, a brief introduction to the classical dynamic mode decomposition (DMD) framework is provided. For details, we refer the reader to [14, 15, 16] and the references therein. Consider the system of time-invariant ordinary differential equations of the form

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) \tag{1}$$

where $\mathbf{x} \in \mathbf{R}^n$ is the state vector and $f : \mathbf{R}^n \to \mathbf{R}^n$ is a nonlinear map ($n \gg 1$). Let the discrete-time representation of (1) be

$$\mathbf{x}_{k+1} = F(\mathbf{x}_k), \tag{2}$$

where $\mathbf{x}_k \in \mathbf{R}^n$ is a high-dimensional state vector sampled at $t_k = k\triangle t$ for $k = 0, \ldots, m$, and F is an unknown map that describes the evolution of the state vector between two subsequent sampling times. The initial condition is defined by $\mathbf{x}(0) = \mathbf{x}_0$. Suppose that the evolution of the high-dimensional state $\mathbf{x}$ is governed by some underlying low-dimensional dynamics. Then, the DMD computes a data-driven linear approximation to the system (2) as follows: the sequential set of data

$$\mathscr{D} = [\mathbf{x}_0, \ldots, \mathbf{x}_m] \tag{3}$$

is arranged into the following two large data matrices

$$X = [\mathbf{x}_0, \ldots, \mathbf{x}_{m-1}] \quad \text{and} \quad Y = [\mathbf{x}_1, \ldots, \mathbf{x}_m]. \tag{4}$$

The goal of the DMD approach is to find a relationship between the future state $\mathbf{x}_{k+1}$ and the current state $\mathbf{x}_k$, given by

$$\mathbf{x}_{k+1} = A\mathbf{x}_k, \tag{5}$$

where $A \in \mathbf{R}^{n\times n}$ is called the DMD operator. The solution of (5) may be expressed simply in terms of the eigenvalues λ_j and eigenvectors ϕ_j of A:

$$\mathbf{x}_k = \sum_{j=1}^{r} \phi_j \lambda_j^k b_j = \Phi\Lambda^k\mathbf{b}, \tag{6}$$

where Φ is the eigenvector matrix of A, Λ is the diagonal matrix of eigenvalues $\Lambda = \mathrm{diag}\{\lambda_i\}$, $\mathbf{b} = \Phi^\dagger\mathbf{x}_0$, and $\Phi^\dagger$ is the Moore–Penrose pseudoinverse of Φ. The parameter r is determined by the low-rank eigendecomposition of matrix A.

1.2 *Augmented DMD (Delay-embedding DMD or Hankel DMD)*

Delay-embedding is also an important technique when the temporal or spectral complexity of a dynamical system exceeds the spatial complexity, for example, in systems characterized by a broadband spectrum or spatially undersampled. In this case, we arrive at a 'short-and-wide', rather than a 'tall-and-skinny', data matrix $\mathscr{D}$, and the standard algorithm fails at extracting all relevant spectral features. Delay-embedding DMD (or Hankel DMD) overcomes several shortcomings of the standard DMD method by extending its capabilities to handle nonlinear dynamics, nonuniformly sampled data, long-term temporal behavior, high-dimensional datasets, and noisy data. This makes it a more versatile and robust technique for dynamic mode decomposition in various applications. The Takens embedding theorem [37] provides a rigorous framework for analyzing the information content of measurements of a nonlinear dynamical system. To implement delay-embedding DMD, given the data sequence $\mathscr{D}$ in (3), we stack s time-shifted copies of the data to form the augmented input matrix. The following Hankel matrix is formed:

$$\mathscr{D}_{aug} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_{m-s+1} \\ \mathbf{x}_2 & \mathbf{x}_3 & \dots & \mathbf{x}_{m-s+2} \\ \dots & \dots & \dots & \dots \\ \mathbf{x}_s & \mathbf{x}_{s+1} & \dots & \mathbf{x}_m \end{pmatrix}, \tag{7}$$

where the applied embedding dimension is s. The augmented data matrix $\mathscr{D}_{aug}$ is then used in place of $\mathscr{D}$ and processed by the standard DMD algorithm. The DMD algorithm prescribed in Equations (3)–(6) is applied to the augmented matrices $X_{aug}, Y_{aug} \in \mathbf{R}^{(n.s)\times(m-s)}$ in place of X and Y, giving eigenvalues Φ_{aug} and modes Λ_{aug}. The first n rows of Φ_{aug} correspond to the current-state DMD modes and are used to forecast $\mathbf{x}(t)$. Arbabi and Mezić [38] have shown the convergence of this time-shifted approach to the eigenfunctions of the Koopman operator. They also illustrated remarkable improvements in the prediction of simple and complex fluid systems. Further examples and theoretical results on delay-embedding and the Hankel viewpoint of Koopman analysis are given by Brunton et al. [39] and Kamb et al. [40]. They demonstrated that linear time-delayed models are an effective and efficient tool to capture nonlinear and chaotic dynamics.

1.3 *Spatio-temporal DMD (ST-DMD)*

The idea behind the spatio-temporal extension of the DMD method is to extend the application range of DMD by implementing the simultaneous capture of both spatial and temporal dynamics. This approach is particularly useful for analyzing complex systems where dynamics evolve both in space and time, such as fluid flows, biological systems, and climate phenomena. In principle, this expansion can be obtained in two ways:

i). Sequential approach. A temporal DMD algorithm is first applied to the snapshot matrix and a spatial DMD algorithm is applied to the spatial modes. Obviously, the order in which temporal and spatial DMDs are applied can be reversed, and the result of the direct and reverse methods is not identical.

ii). Parallel approach. Reduced SVD is first applied to the snapshot matrix D, and then, spatial and temporal DMD algorithms are applied to the rescaled left and right singular vector matrices.

2 Sequential STDMD algorithms

Our goal in this study is to present a delayed coordinate modification of the sequential STDMD method. A corresponding modification for the parallel STDMD method can be seen in [36]. In this section, we will first review the mathematical framework of *sequential STDMD* and then present its modification with delay-embedded coordinates.

2.1 *Sequential STDMD*

In the following, we provide a detailed mathematical description of the sequential STDMD approach, which we will extend by delayed coordinates in the next paragraph. Sequential STDMD involves decomposing spatio-temporal data sequentially along the temporal axis, capturing both spatial and temporal dynamics separately. This approach enables the identification of spatial structures evolving over time and their corresponding temporal dynamics. For conventional DMD, the temporal information (temporal growth rate and angular frequency) is explicitly included in the eigenvalue matrix Λ, whereas the spatial information (spatial growth rate and wavenumber) is implicitly hidden in the dynamic mode matrix F. Therefore, our goal is to decompose dynamic modes in a certain way to obtain spatial information.

Let us apply the standard DMD method described above to the input data $\mathscr{D}$ specified in (3), which results in temporal DMD expansion

$$\mathbf{x}_k = \Phi \Lambda^k \mathbf{b}, \tag{8}$$

where $\Phi = UW$ is the matrix of DMD modes, Λ is the matrix of DMD eigenvalues and $\mathbf{b}$ is the vector of amplitudes. Matrix U in expression of Φ is from the reduced SVD decomposition $X = U\Sigma V^*$, where $U \in \mathbf{R}^{m\times r}, \Sigma \in \mathbf{R}^{r\times r}$ and $V \in \mathbf{R}^{n\times r}$; see [14, 15].

It is straightforward to show that snapshots data $\mathscr{D}$ has the following equivalent expression:

$$\mathscr{D} = \Phi\,\mathrm{diag}\{b_i\} V_{and}(\lambda)\,, \tag{9}$$

where $V_{and}(\lambda)$ is a Vandermonde matrix

$$V_{and}(\lambda) = \begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_1^m \\ 1 & \lambda_2 & \dots & \lambda_2^m \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_r & \dots & \lambda_r^m \end{pmatrix}. \tag{10}$$

This demonstrates that the temporal evolution of the dynamic modes is governed by the Vandermonde matrix, which is determined by the r complex eigenvalues λ_i of reduced DMD operator $\tilde{A} = U^*AU$ containing information about the underlying temporal frequencies and growth/decay rates.

Note that the spatial information, as spatial growth rate and wavenumber, of the dynamic in consideration is implicitly hidden in the dynamic mode matrix Φ. We can use the row-vectors of the DMD mode matrix Φ to get spatial expansion similar to (6). Let us denote

$$\bar{\mathscr{D}} = \Phi^T = [\bar{\mathbf{y}}_0, \dots, \bar{\mathbf{y}}_n]\,, \tag{11}$$

where $\bar{\mathbf{y}}_i$ is the i-th column-vector of $\bar{\mathscr{D}}$. Note that the matrix $\bar{\mathscr{D}}$ has full rank r. Applying the standard DMD approach to data $\bar{\mathscr{D}}$, we get the following expansion according to (6):

$$\bar{\mathbf{y}}_k = \bar{\Phi}\bar{\Lambda}^k \bar{\mathbf{b}}, \tag{12}$$

where $\bar{\Phi}$ is the eigenvector matrix, $\bar{\Lambda} = \operatorname{diag}\{\bar{\lambda}_i\}$ is the diagonal matrix of associated eigenvalues of the corresponding DMD operator, and $\bar{\mathbf{b}} = \bar{\Phi}^{-1}\bar{\mathbf{y}}_0$.

Then, for the full-data matrix $\mathscr{D}$, we get the following matrix-form presentation:

$$\mathscr{D} = V_{and}^T\left(\bar{\lambda}\right)\operatorname{diag}\left\{\bar{b}_i\right\}\Psi\operatorname{diag}\{b_i\}V_{and}(\lambda), \tag{13}$$

where $r \times r$ matrix

$$\Psi = \bar{\Phi}^T \tag{14}$$

is the matrix of *spatio-temporal DMD modes*.

The following algorithm summarizes the steps for sequential STDMD:

Algorithm 1: Sequential STDMD

1. Perform the standard DMD approach to data set $\mathscr{D}$ and compute DMD modes, eigenvalues and amplitudes: Φ, Λ and $\mathbf{b}$.
2. Define the spatial data matrix as transposed DMD modes: $\bar{\mathscr{D}} = \Phi^T$.
3. Perform the standard DMD approach to data set $\bar{\mathscr{D}}$ and compute DMD modes, eigenvalues and amplitudes: $\bar{\Phi}$, $\bar{\Lambda}$ and $\bar{\mathbf{b}}$.
4. Compute the matrix of spatio-temporal DMD modes $\Psi = \bar{\Phi}^T$.

For the reconstruction of snapshots in $\mathscr{D}$, we get similar to (6) expression

$$x_s^{(k)} = \sum_{i,j=1}^{r} \psi_{ij}\bar{\lambda}_i^s \bar{b}_i \lambda_j^k b_j, \tag{15}$$

where $x_s^{(k)}$ is the s-th coordinate of state $\mathbf{x}_k$.

We also can convert discrete-time eigenvalues $\bar{\lambda}_i$ and λ_j to the continuous time eigenvalues $\bar{\alpha}_i$ and α_j, respectively. The conversion formulas are

$$\alpha_j = \frac{\ln\lambda_j}{\triangle t} \quad \text{and} \quad \bar{\alpha}_i = \frac{\ln\bar{\lambda}_i}{\triangle x} \quad \text{for} \quad i,j = 1,\ldots,r. \tag{16}$$

Let us denote

$$\alpha_j = \delta_j + i\omega_j \quad \text{and} \quad \bar{\alpha}_i = \nu_i + i\kappa_i, \tag{17}$$

where δ_j are the *temporal growth rates*, ω_j are the (temporal) *frequencies*, ν_i are the *spatial growth rates*, and the (spatial) *wavenumbers* are κ_i. The the corresponding continuous time expansion of (15) is following:

(18) $$x_s^{(k)} = \sum_{i,j=1}^{r} \psi_{ij}\bar{b}_i b_j e^{(\delta_j+i\omega_j)t+(\nu_i+i\kappa_i)x}.$$

2.2 *Augmented sequential STDMD*

As already mentioned, traditional DMD approaches are limited in their ability to capture the full complexity of nonlinear and non-stationary systems, particularly when dealing with high-dimensional and noisy datasets. Due to the fact that in Algorithms 1 the standard DMD method is applied sequentially, it inherits the disadvantages of the DMD method. To address these limitations, we will propose an extension of STDMD algorithm using the delay-embedding approach described in Section1. This approach redesigns the input data of the system, creating new state variables. However, the introduction of the new variables is made at the expense of reducing the number of samples in the training data set. Hence, the number of these new variables (number of rows in the Hankel matrix), in (7), has to be a balance between the ability to detect dominant modes and the accuracy of the estimated model. The following algorithm (Algorithm 2) provides a step-by-step implementation of augmented sequential spatio-temporal DMD:

Algorithm 2: Augmented sequential STDMD

1. Perform augmented DMD approach to data set $\mathscr{D}$ and compute temporal DMD modes, eigenvalues and amplitudes: Φ, Λ and $\mathbf{b}$.
2. Define the spatial data matrix as transposed DMD modes: $\bar{\mathscr{D}} = \Phi^T$.
3. Perform augmented DMD approach to data set $\bar{\mathscr{D}}$ and compute spatial DMD modes, eigenvalues and amplitudes: $\bar{\Phi}$, $\bar{\Lambda}$ and $\bar{\mathbf{b}}$.
4. Compute the matrix of spatio-temporal DMD modes $\Psi = \bar{\Phi}^T$.

The STDMD approach, augmented with delay-embedding, offers enhanced computational efficiency compared to STKD. By augmenting the dataset with delayed observations, the analysis captures underlying dynamics more effectively, reducing the impact of noise on mode identification and reconstruction. Overall, the delay-embedding STDMD enhances the accessibility and usability of the proposed approaches, making them more practical and widely applicable to researchers and practitioners in various fields.

For the reconstruction of snapshots in $\mathscr{D}$, we get similar to (15) expression

(19) $$x_s^{(k)} = \sum_{i=1}^{r_s}\sum_{j=1}^{r_t} \psi_{ij}\bar{\lambda}_i^s\bar{b}_i\lambda_j^k b_j,$$

where $x_s^{(k)}$ is the s-th coordinate of state $\mathbf{x}_k$. In (19) r_t denotes the number of temporal DMD modes determined at step 1 of Algorithm 2, and r_s denotes the number of spatial DMD modes determined at step 3 of Algorithm 2.

The corresponding continuous time expansion of (19) is following:

$$x_s^{(k)} = \sum_{i=1}^{r_s} \sum_{j=1}^{r_t} \psi_{ij} \bar{b}_i b_j e^{(\delta_j + i\omega_j)t + (\nu_i + i\kappa_i)x}. \tag{20}$$

Expression (20), compared to (18), uses a larger number of temporal and spatial DMD modes, as a result of delayed coordinates approach in Algorithm 2.

3 Numerical examples

Here, we will illustrate the introduced approach sequential augmented STDMD (Algorithm 2). The considered examples are benchmark, and through them, we illustrate the ability of the proposed scheme to accurately calculate spatiotemporal DMD modes and eigenvalues, including spatial wavenumbers and growth rates and temporal frequencies and growth rates.

Example 1: Superposition of three sine waves with exponential decay

Let us consider a wave-field that is a combination of three sine waves with different wavenumbers and frequencies, and decaying exponentially in time. Consider a spatio-temporal signal:

$$u(x,t) = e^{-at} \left(\sin(k_1 x - \omega_1 t) + \sin(k_2 x - \omega_2 t) + \sin(k_1 x - \omega_3 t) \right), \tag{21}$$

where:

- k_1 and k_2 are the wavenumbers of the first two waves (for the third wave we use the same wavenumber as for the first wave);
- ω_1, ω_2 and ω_3 are the angular frequencies of the three waves;
- a is the exponential damping factor.

We set the parameters to: $k_1 = 2\pi, k_2 = 3\pi$, $\omega_1 = 2\pi, \omega_2 = 4\pi, \omega_3 = 6\pi$ and $a = 0.5$. In order to apply the augmented sequential STDMD method, we discretize x and t in the sampled intervals $0 \le x \le 2$ and $0 \le t \le 2$, using steps $\triangle x = \triangle t = 0.01$. Generated data are 201×201, with rank one, which yields unsatisfactory results with the pure temporal DMD method.

The 3D visualization of signal is depicted in Figure 1 (left) and the color map is in Figure 1 (right). This pattern is obtained with a spectral spatial and temporal complexity of 4 and 6, respectively. This is because it involves 4 wavenumbers: $\pm k_1$ and $\pm k_2$, and 6 frequencies: $\pm\omega_1, \pm\omega_2$ and $\pm\omega_3$.

Performing augmented sequential STDMD (Algorithm 2), with time-delaying index 2 and spatial-delaying index 1, we identify all the correct wavenumbers and frequencies. Figure 2 depicts the growth rate–frequency and amplitude–frequency diagrams. Using expression (15) we obtain an approximation of the dynamics data and reconstruct the snapshots with a relative RMS error: 1.8933×10^{-13}.

We note that in this example it was enough to use lagging coordinates only in the first part of Algorithm 2 (in step 1), and in the second part (step 3) the standard DMD method was used.

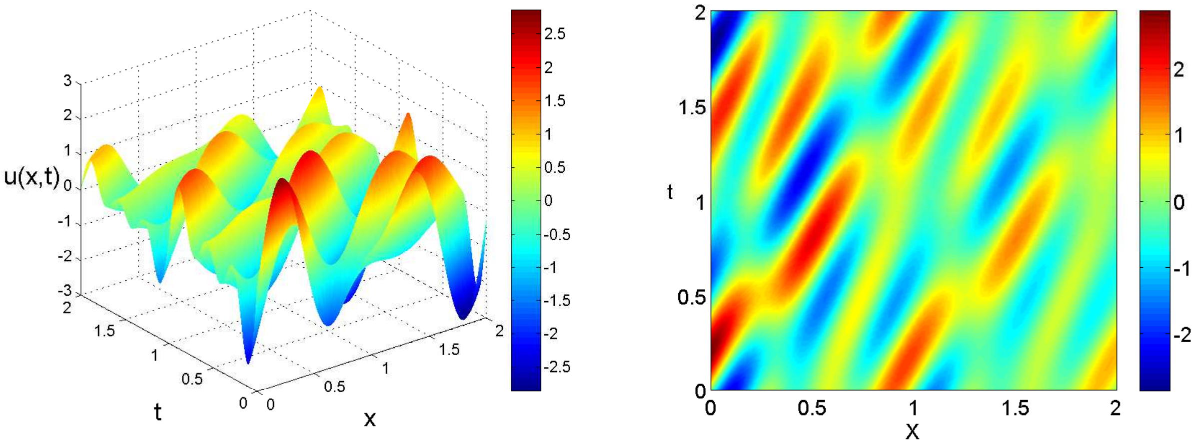

Figure 1: *3D visualization (left) and color-map (right) of spatio-temporal signal (21).*

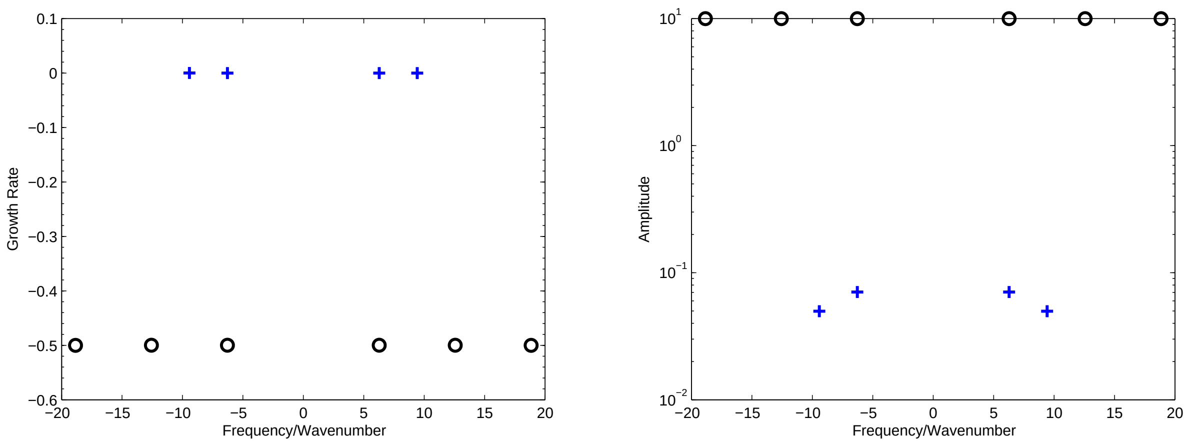

Figure 2: *Left: Spatial growth rate-wavenumber ('+') and temporal growth rate-frequency ('o'); Right: Spatial amplitudes-wavenumber ('+') and temporal amplitudes-frequency ('o').*

Example 2: Spatiotemporal signal of a traveling-wave

The toy model considered here was introduced in [41] and represents a combination of traveling waves, defined as

$$u(x,t) = \left(\frac{1}{2} + \sin(x)\right)\left(2\cos(k_1 x - \omega_1 t) + \frac{1}{2}\cos(k_2 x - \omega_2 t)\right), \tag{22}$$

with

$$k_1 = 2, k_2 = 10, \omega_1 = 2\pi \text{ and } \omega_2 = \sqrt{2}.$$

The model is visualized in Figure 3, see also [35]. This pattern is obtained with the following 12 wavenumber/frequency pairs:

$$\begin{aligned} &\pm(k_1,\omega_1), \pm(k_1-1,\omega_1), \pm(k_1+1,\omega_1), \\ &\pm(k_2,\omega_2), \pm(k_2-1,\omega_2), \pm(k_2+1,\omega_2). \end{aligned} \tag{23}$$

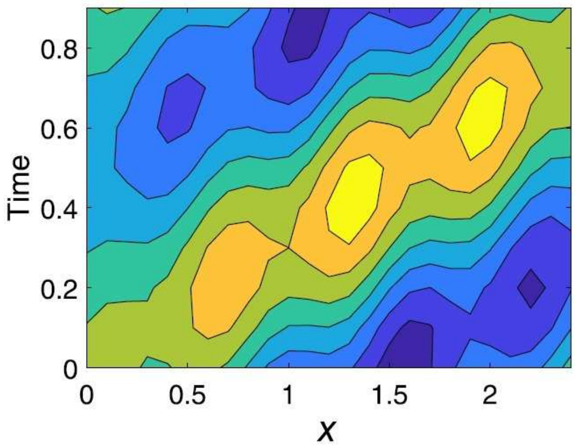

Figure 3: *Visualization of model (22).*

Note that the pattern is temporally quasiperiodic, because the two involved frequencies, ω_1 and ω_2, are incommensurable, but it is spatially periodic, with period equal to 2π (wavenumber equal to 1).

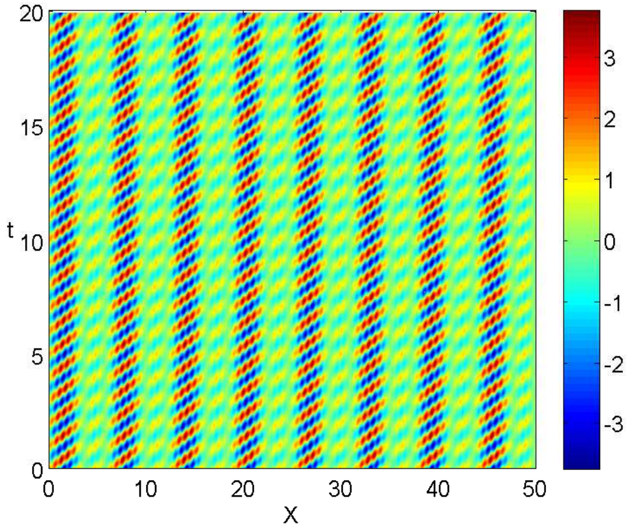

Figure 4: *Color map for the model (22).*

We apply the augmented sequential STDMD method in the training spatio-temporal set $0 \leq x \leq 50$, $0 \leq t \leq 20$, considering 500 equispaced values of x and 200 equispaced values of t, with $\triangle x = \triangle t = 0.1$. The color map of the model is shown in Figure 4.

Performed augmented sequential STDMD (Algorithm 2), with time-delaying index 1 and spatial-delaying index 3. With this selection, Algorithm 2 recognizes the exact wavenumber/frequency pairs given in (23); see the dispersion diagram in Fig. 5.

Figure 6 depicts the growth rate–frequency and amplitude–frequency diagrams. We reconstructs the model with a relative RMS error: 1.2×10^{-11}. The results are identical to those in [35], where the spatio-temporal Koopman decomposition (STKD) method is applied to the same example and input data.

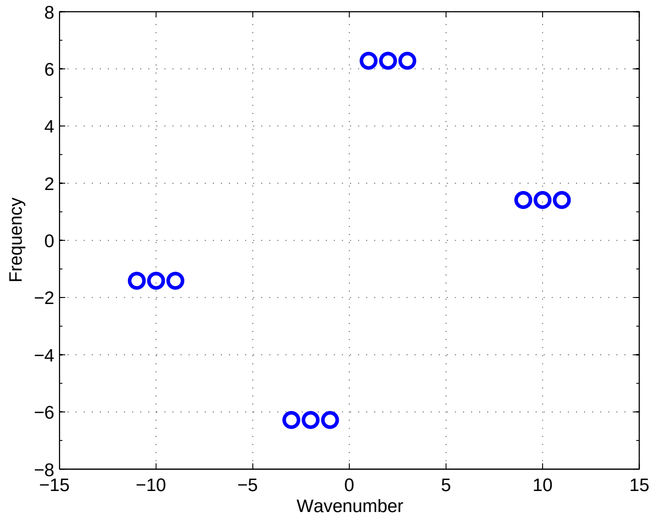

Figure 5: *Dispersion diagram resulting from applying the augmented sequential STDMD method to the model (22).*

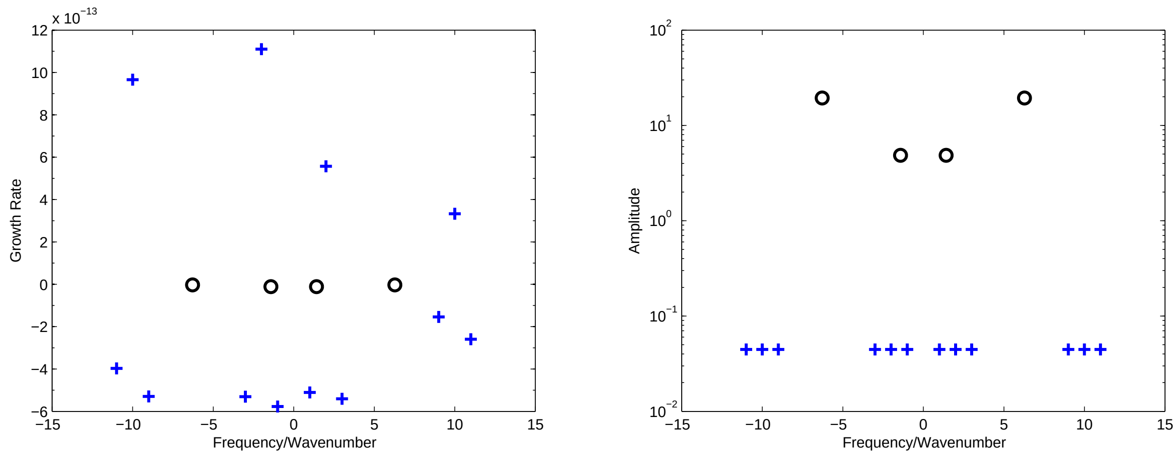

Figure 6: *Left: Spatial growth rate-wavenumber ('+') and temporal growth rate-frequency ('o'); Right: Spatial amplitudes-wavenumber ('+') and temporal amplitudes-frequency ('o').*

4 Conclusion

The purpose of this study was to introduce a new modification of spatio-temporal DMD method by incorporating delay-embedding techniques. We refer the new echnique augmented sequential STDMD method. The matrix representations underlying this technique is provided, highlighting their respective computational frameworks for analyzing spatiotemporal data.

We have demonstrated the performance of the presented algorithm with illustrative numerical examples. The numerical results show that the introduced algorithm is an alternative to the parallel deley-embedded STDMD and can be used in various fields of application.

REFERENCES:

[1] Schmid, P.J. and Sesterhenn, J., Dynamic mode decomposition of numerical and experimental data, in: 61st Annual Meeting of the APS Division of Fluid Dynamics, American Physical Society, November 2008.

[2] Grosek, J., Kutz., J.N., Dynamic Mode Decomposition for Real-Time Background/Foreground Separation in Video, arXiv: 1404.7592.

[3] Proctor, J.L., and Eckhoff, P.A., Discovering dynamic patterns from infectious disease data using dynamic mode decomposition, International health, 7 (2015), 139–145.

[4] Berger, E., Sastuba, M., Vogt, D., Jung, B., and Amor, H.B., Estimation of perturbations in robotic behavior using dynamic mode decomposition, Journal of Advanced Robotics, 29 (2015), 331–343.

[5] Brunton, B.W., Johnson, L.A., Ojemann, J.G., and Kutz, J.N., Extracting spatial-temporal coherent patterns in large-scale neural recordings using dynamic mode decomposition, Journal of Neuroscience Methods, 258 (2016), 1–15.

[6] Mann, J., and Kutz, J.N., Dynamic mode decomposition for financial trading strategies, Quantitative Finance (2016), 1–13.

[7] Cui, L., Long, W., Trading strategy based on dynamic mode decomposition: Tested in Chinese stock market, Physica A: Statistical Mechanics and its Applications, Elsevier. 461(C), (2016), 498–508.

[8] Kuttichira, D.P., Gopalakrishnan, E.A., Menon, V.K., and Soman, K.P., Stock price prediction using dynamic mode decomposition, in: 2017 International Conference on Advances in Computing, Communications and Informatics (ICACCI), Udupi, India, (2017), 55–60.

[9] Schmid, P.J., Dynamic mode decomposition of numerical and experimental data, J. Fluid Mech. 656, (2010), 5–28.

[10] Seena, A., and Sung, H.J., Dynamic mode decomposition of turbulent cavity flows for selfsustained oscillations, Int. J. Heat Fluid Fl., 32(6), (2011), 1098–1110.

[11] Rowley, C.W., Mezić, I., Bagheri, S., Schlatter, P., and Henningson, D.S., Spectral analysis of nonlinear flows, J. Fluid Mech., 641, (2009), 115–127.

[12] Schmid, P.J., Application of the dynamic mode decomposition to experimental data, Exp. Fluids, 50(4), (2011), 1123–1130.

[13] Mezić, I., Analysis of fluid flows via spectral properties of the Koopman operator, Annu. Rev. Fluid Mech., 45, (2013), 357–378.

[14] Tu, J.H., Rowley, C.W., Luchtenburg, D.M., Brunton, S.L., Kutz, J.N., On dynamic mode decomposition: Theory and applications, Journal of Computational Dynamics, 1 (2), (2014), 391–421.

[15] Kutz, J.N., Brunton, S.L., Brunton, B.W., and Proctor, J., Dynamic Mode Decomposition: Data-Driven Modeling of Complex Systems, SIAM 2016, ISBN 978-1-611-97449-2, (2016).

[16] Brunton, S.L., Budišić, M., Kaiser, E., and Kutz, J.N., Modern Koopman Theory for Dynamical Systems, SIAM Review, 64:2, (2022), 229–340.

[17] Bai, Z., Kaiser, E., Proctor, J.L., Kutz, J.N., Brunton, S.L., Dynamic Mode Decomposition for Compressive System Identification, AIAA Journal, 58 (2), (2020), 561–574.

[18] Nedzhibov, G., Dynamic Mode Decomposition: a new approach for computing the DMD modes and eigenvalues, Ann. Acad. Rom. Sci. Ser. Math. Appl.14 (1–2), (2022).

[19] Le Clainche, S., Vega, J.M., Soria, J., Higher order dynamic mode decomposition of noisy experimental data: The flow structure of a zero-net-mass-flux jet, Experimental Thermal and Fluid Science, 88, (2017).

[20] Anantharamu, S., Mahesh, K., A parallel and streaming Dynamic Mode Decomposition algorithm with finite precision error analysis for large data, Journal of Computational Physics 380, (2019).

[21] Sayadi, T., Schmid, P.J.,Parallel data-driven decomposition algorithm for large-scale datasets: with application to transitional boundary layers, Theor. Comput. Fluid Dyn. 30, (2016), 415–428

[22] Maryada, K.R., Norris, S.E., Reduced-communication parallel dynamic mode decomposition, Journal of Computational Science 61, (2022).

[23] Li, B., Garicano-Menaab, J., Valero, E., A dynamic modedecomposition technique for the analysis of non-uniformly sampled flow data, Journal of Computational Physics 468, (2022).

[24] Smith, E., Variansyah, I., McClarren, R., Variable Dynamic Mode Decomposition for Estimating Time Eigenvalues in Nuclear Systems. https://arxiv.org/abs/2208.10942v1

[25] Jovanović, M.R., Schmid, P.J., Nichols, J.W., Sparsity-promoting dynamic mode decomposition, Physics of Fluids 26, (2014).

[26] Guéniat, F., Mathelin, L., Pastur, L.R., A dynamic mode decomposition approach for large and arbitrarily sampled systems, Phys Fluids 27, (2014).

[27] Cassamo, N., van Wingerden, J.W., On the Potential of Reduced Order Models for Wind Farm Control: A Koopman Dynamic Mode Decomposition Approach, Energies 13, (2020), 6513.

[28] Ngo, T.T., Nguyen, V., Pham, X.Q., Hossain, M.A., Huh, E.N., Motion Saliency Detection for Surveillance Systems Using Streaming Dynamic Mode Decomposition, Symmetry 12, (2020), 1397.

[29] Babalola, O.P., Balyan, V., WiFi Fingerprinting Indoor Localization Based on Dynamic Mode Decomposition Feature Selection with Hidden Markov Model, Sensors 21, (2021), 6778.

[30] Lopez-Martin, M., Sanchez-Esguevillas, A., Hernandez-Callejo, L., Arribas, J.I., Carro, B., Novel Data-Driven Models Applied to Short-Term Electric Load Forecasting, Appl. Sci. 11, (2021), 5708.

[31] Surasinghe, S., Bollt, E.M., Randomized Projection Learning Method for Dynamic Mode Decomposition, Mathematics 9, (2021), 2803. https://doi.org/10.3390/math9212803

[32] Li, C.Y., Chen, Z., Tse, T.K., A parametric and feasibility study for data sampling of the dynamic mode decomposition: range, resolution, and universal convergence states, Nonlinear Dyn 107, (2022), 3683–3707 .

[33] Mezić, I., On Numerical Approximations of the Koopman Operator" Mathematics 10, (2022), 1180.

[34] Sharma, A.S., Mezić, I., McKeon, B.J., Correspondence between Koopman mode decomposition, resolvent mode decomposition, and invariant solutions of the Navier-Stokes equations, Phys. Rev. Fluids 1, (2016), 032402.

[35] Clainche, S.L., Vega, J.M., Spatio-Temporal Koopman Decomposition, J Nonlinear Sci 28, (2018), 1793–1842.

[36] Nedzhibov, G., Delay-Embedding Spatio-Temporal Dynamic Mode Decomposition, Mathematics 12(5):762, (2024).

[37] Takens, F., Detecting strange attractors in turbulence, in Dynamical Systems and Turbulence, Lecture Notes in Math. 898, Springer, Berlin, (1981).

[38] Arbabi, H., and Mezić, I., Ergodic Theory, Dynamic Mode Decomposition, and Computation of Spectral Properties of the Koopman Operator, SIAM Journal on Applied Dynamical Systems 16 (2017), 2096–2126,

[39] Brunton, S.L., Brunton, B.W., Proctor, J.L., Kaiser, E., and Kutz, J.N., Chaos as an intermittently forced linear system, Nature Commun. 8 (2017), 19.

[40] Kamb, M., Kaiser, E., Brunton, S.L., and Kutz, J.N., Time-delay observables for Koopman: Theory and applications, SIAM J. Appl. Dyn. Syst. 19 (2020), 886–917.

[41] Giannakis, D., Slawinska, J., Zhao, Z., Spatiotemporal Feature Extraction with Data-Driven Koopman Operators, JMLR: Workshop and Conference Proceedings 44, (2015), 103–115.