ITERATIVE METHODS FOR FINDING A POSITIVE DEFINITE SOLUTION OF THE MATRIX EQUATION $X - A^*XA - B^*X^{-1}B = I$*

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ABSTRACT: *In this paper we consider modifications of two iterative methods for finding a positive definite solution of the matrix equation $X - A^*XA - B^*X^{-1}B = I$, proposed in [1, 2]. We prove the convergence of the considered methods and obtain a sufficient condition for the uniqueness of the solution. Finally, we propose some numerical examples that illustrate the convergence behavior of the considered methods.*

KEYWORDS: *Matrix equation, positive definite solution, iterative methods.*

1 Introduction

In this paper we consider iterative methods for finding a positive definite solution of the nonlinear matrix equation

$$X - A^*XA - B^*X^{-1}B = I, \tag{1}$$

where A, B are $n \times n$ complex matrices, $I\,(I_n)$ is the $n \times n$ identity matrix, X is unknown matrix, and A^* denotes the conjugate transpose of A. This equation contains two well-known matrix equations – the Stein's equation $X - A^*XA = I$ and the equation $X - B^*X^{-1}B = I$ [3]. Moreover, Eq. (1) is an extension of the Gao's equation $X - A^*X^pA - B^*X^{-q}B = I$ with $0 < p, q < 1$ [4] for $p = q = 1$.

Eq. (1) has been introduced by Ali in [1], where the iterative method

$$\begin{cases} X_0 = I, \;\; Y_0 = \beta I, \;\; \beta > 1, \\ X_{k+1} = I + A^*X_kA + B^*Y_k^{-1}B, \quad k = 0, 1, \ldots \\ Y_{k+1} = I + A^*Y_kA + B^*X_k^{-1}B \end{cases} \tag{2}$$

for computing a positive definite solution is proposed. It has been shown that the sequences $\{X_k\}$ and $\{Y_k\}$ defined by (2) with $\beta \geq (1 + \|B\|^2)/(1 - \|A\|^2)$, have the following properties

$$X_0 \leq X_1 \leq \cdots \leq X_k \leq Y_k \leq \cdots \leq Y_1 \leq Y_0 \tag{3}$$

and converge to a unique positive definite solution of Eq. (1) under condition $\|A\|^2 + \|B\|^2 < 1$. Moreover, the necessary conditions for existence of a positive definite solution and its lower bound have been obtained: If X_+ is a positive definite solution of Eq. (1), then $\rho(A) < 1$, $\rho(X_+^{-1}B) < 1$, and $X_+ \geq M$, where M is a unique solution of the equation $X - A^*XA = I$ (see [1, Theorem 2.1.]).

In [5], we have derived that $\rho(A) < 1$ is a necessary and sufficient condition for the existence of a positive definite solution of Eq. (1), an upper bound of all the solutions has been obtained: all positive definite solutions are in $[M, N]$, where M and N are the unique solutions of the equations $X - A^*XA = I$ and $X - A^*XA = I + B^*M^{-1}B$, respectively (see [5, Theorem 2.]). Moreover, the basic fixed point iteration (BFPI):

$$Z_{k+1} = I + A^*Z_kA + B^*Z_k^{-1}B, \quad k = 0, 1, \ldots, \;\; Z_0 \in [M, N], \tag{4}$$

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have been proposed. The sequences $\{Z_k\}$, $\{X_k\}$ and $\{Y_k\}$ defined by (4) and (2) have the following properties $X_k \le Z_k \le Y_k$, $k = 0, 1, \ldots$.

Ali and Hasanov [2] have considered following partially inverse free variant of the iterative method (2):

$$\begin{cases} X_0 = M, \ Y_0 = N, \ (\text{or } X_0 = I, Y_0 = \beta I), \ V_0 = Y_0^{-1}, \\ V_{k+1} = V_k(2I - Y_k V_k), \\ X_{k+1} = I + A^* X_k A + B^* V_{k+1} B, \quad k = 0, 1, \ldots \\ Y_{k+1} = I + A^* Y_k A + B^* X_k^{-1} B, \end{cases} \tag{5}$$

and have proved the following properties of (5):

$$\begin{cases} X_k \le X_{k+1} \le Y_{k+1} \le Y_k, \quad V_k \le V_{k+1} \le Y_k^{-1}, \quad k = 0, 1, \ldots, \\ \lim_{k\to\infty} X_k = \bar{X} \le \bar{Y} = \lim_{k\to\infty} Y_k, \quad \lim_{k\to\infty} V_k = \bar{Y}^{-1}. \end{cases} \tag{6}$$

Motivated by above investigations [1, 2, 5, 4], we study Eq. (1) and propose two iterative methods that are modifications of methods (2) and (5).

Throughout this paper, we denote by $\rho(A)$ the spectral radius, by $\|A\|$ the spectral norm $\left(\|A\| = \sqrt{\rho(A^*A)}\right)$, by $A \otimes B$ the Kronecker product of the matrices A and B, and $vec(A)$ is a vector, defined by $vec(A) = (a_1^T, \ldots, a_n^T)^T$, where a_i is i-th column of the matrix A. $A > 0$ ($A \ge 0$) means that A is a positive definite (semidefinite) matrix. If $A - B > 0$ (or $A - B \ge 0$) we write $A > B$ (or $A \ge B$). For $B \ge A > 0$ we use $[A, B]$ to denote the set of matrices $\{X : A \le X \le B\}$.

2 Iterative methods

We note that the iterative method (2) can be used with $X_0 = M$ and $Y_0 = N$. In this case, for method (2) we have

$$Y_1 = I + A^* Y_0 A + B^* X_0^{-1} B = I + A^* N A + B^* M^{-1} B = N = Y_0.$$

We consider a modification of the iterative method (2) by replacing X_k with X_{k+1} in the formula for calculating Y_{k+1}.

$$\begin{cases} X_0 = M, \ Y_0 = N, \\ X_{k+1} = I + A^* X_k A + B^* Y_k^{-1} B, \quad k = 0, 1, \ldots \\ Y_{k+1} = I + A^* Y_k A + B^* X_{k+1}^{-1} B. \end{cases} \tag{7}$$

For method (7), we have

$$\begin{aligned} X_1 &= I + A^* M A + B^* N^{-1} B = M + B^* N^{-1} B \ge M = X_0, \\ Y_1 &= I + A^* N A + B^* X_1^{-1} B \le I + A^* N A + B^* M^{-1} B = N = Y_0, \end{aligned}$$

and by induction it follows

$$X_0 \le X_1 \le \cdots \le X_k \le Y_k \le \cdots \le Y_1 \le Y_0. \tag{8}$$

From above properties, we conclude that the sequences $\{X_k\}$ and $\{Y_k\}$ are convergent and

$$\lim_{k\to\infty} X_k = \hat{X} \le \hat{Y} = \lim_{k\to\infty} Y_k.$$

To compare the matrix sequences generated by methods (2) and (7), we introduce the following notation: $\{X_k\}$, $\{Y_k\}$ and $\{X'_k\}$, $\{Y'_k\}$ are sequences generated by (2) with $X_0 \le M, Y_0 \ge N$ (for example $X_0 = I, Y_0 = \beta I$) and $X'_0 = M, Y'_0 = N$, respectively, and $\{X''_k\}$, $\{Y''_k\}$ are sequences generated by (7) with $X''_0 = M, Y''_0 = N$. Then we have

$$\begin{aligned} &X_0 \le X'_0 = X''_0 \ \text{ and } \ Y''_0 = Y'_0 \le Y_0 \\ &X_1 \le X'_1 = X''_1 \ \text{ and } \ Y''_1 \le Y'_1 \le Y_1 \\ &X_k \le X'_k \le X''_k \ \text{ and } \ Y''_k \le Y'_k \le Y_k, \quad k = 2,3,\ldots. \end{aligned}$$

Therefore, $\|Y''_k - X''_k\| \le \|Y'_k - X'_k\| \le \|Y_k - X_k\|$, $k = 0,1,\ldots$.

With the following result, we improve the sufficient condition $\|A\|^2 + \|B\|^2 < 1$ [1] for the uniqueness of the positive definite solution to Eq. (1).

Theorem 1. *Let $\rho(A) < 1$ and the sequences $\{X_k\}$ and $\{Y_k\}$ be generated by iterative method (7). Then*

(i) Eq. (1) has a positive definite solution X_+ and $X_+ \in [\hat{X}, \hat{Y}]$, where $\hat{X} = \lim_{k\to\infty} X_k$ and $\hat{Y} = \lim_{k\to\infty} Y_k$;

(ii) $\|Y_{k+1} - X_{k+1}\| \le \|A\|^2 \|Y_k - X_k\| + \|X_{k+1}^{-1}B\| \|Y_k^{-1}B\| \|Y_k - X_{k+1}\|$, $k = 0,1,\ldots$;

(iii) for all $\varepsilon > 0$, we have $\|Y_{k+1} - X_{k+1}\| \le \left[\|A\|^2 + (\|\hat{X}^{-1}B\| + \varepsilon)^2\right] \|Y_k - X_k\|$ for all k large enough.

Moreover, if $\|A\|^2 + \|\hat{X}^{-1}B\|^2 < 1$, then $X_+ = \hat{X} = \hat{Y}$ is a unique positive definite solution of Eq. (1) and the sequences $\{X_k\}$ and $\{Y_k\}$ converge linearly to X_+.

Proof. (i): Under condition $\rho(A) < 1$, the sequences $\{X_k\}$ and $\{Y_k\}$ are well defined, have following properties $X_k \le X_{k+1} \le Y_{k+1} \le Y_k$, $k = 0,1,\ldots$ (see (8)), and ere convergent. Let $\hat{X} = \lim_{k\to\infty} X_k$ and $\hat{Y} = \lim_{k\to\infty} Y_k$. From above properties, we have $\hat{X} \le \hat{Y}$. According to $\rho(A) \le \|A\| < 1$ and [5, Theorem 2.], we have that Eq. (1) has a positive definite solution X_+, and $X_+ \in [M,N]$, where M and N are the unique solutions of the equations $X - A^*XA = I$ and $X - A^*XA = I + B^*M^{-1}B$, respectively, i.e. $X_+ \in [X_0, Y_0]$. We will prove by induction that $X_+ \in [X_k, Y_k]$ for every $k = 1,2,\ldots$, using the technique from [8].

Assume that $X_+ \in [X_k, Y_k]$ for fixed k. Using formulae (7) and induction assumption, we obtain

$$\begin{aligned} X_{k+1} &= I + A^*X_kA + B^*Y_k^{-1}B \le I + A^*X_+A + B^*X_+^{-1}B = X_+, \\ Y_{k+1} &= I + A^*Y_kA + B^*X_{k+1}^{-1}B \ge I + A^*X_+A + B^*X_+^{-1}B = X_+, \end{aligned}$$

i.e., $X_+ \in [X_{k+1}, Y_{k+1}]$. Hence, $X_+ \in [X_k, Y_k]$ for $k = 0,1,\ldots$ and $X_+ \in [\hat{X}, \hat{Y}]$.

By subtracting the equations in (7), we obtain

$$\begin{aligned} Y_{k+1} - X_{k+1} &= A^*(Y_k - X_k)A + B^*(X_{k+1}^{-1} - Y_k^{-1})B \\ &= A^*(Y_k - X_k)A + B^*X_{k+1}^{-1}(Y_k - X_{k+1})Y_k^{-1}B \\ &= A^*(Y_k - X_k)A + B^*X_{k+1}^{-1}(Y_k - X_{k+1})X_{k+1}^{-1}B \\ &\quad - B^*X_{k+1}^{-1}(Y_k - X_{k+1})Y_k^{-1}(Y_k - X_{k+1})X_{k+1}^{-1}B \\ &\le A^*(Y_k - X_k)A + B^*X_{k+1}^{-1}(Y_k - X_{k+1})X_{k+1}^{-1}B \end{aligned}$$

from where, we have

$$\begin{aligned}\|Y_{k+1}-X_{k+1}\| &\le \|A^*(Y_k-X_k)A\| + \|B^*X_{k+1}^{-1}(Y_k-X_{k+1})Y_k^{-1}B\| \\ &\le \|A\|^2\|Y_k-X_k\| + \|X_{k+1}^{-1}B\|\|Y_k^{-1}B\|\|Y_k-X_{k+1}\|\end{aligned}$$

and

$$\begin{aligned}\|Y_{k+1}-X_{k+1}\| &\le \|A\|^2\|Y_k-X_k)\| + \|X_{k+1}^{-1}B\|^2\|Y_k-X_{k+1}\| \\ &\le \left[\|A\|^2 + \|X_{k+1}^{-1}B\|^2\right]\|Y_k-X_k\|.\end{aligned}$$

Since $X_k^{-1} \to \hat{X}^{-1}$ $(X_k \to \hat{X})$, for all $\varepsilon > 0$ there exists an index ν_ε such that for $k > \nu_\varepsilon$ the condition $\|X_k^{-1} - \hat{X}^{-1}\| < \varepsilon/\|B\|$ holds. Then, from

$$\|X_{k+1}^{-1}B\| = \|(\hat{X}^{-1} + X_{k+1}^{-1} - \hat{X}^{-1})B\| \le \|\hat{X}^{-1}B\| + \|X_{k+1}^{-1} - \hat{X}^{-1}\|\|B\|,$$

it follows that $\|X_{k+1}^{-1}B\| < \|\hat{X}^{-1}B\| + \varepsilon$ and $\|Y_{k+1}-X_{k+1}\| \le \left[\|A\|^2 + (\|\hat{X}^{-1}B\| + \varepsilon)^2\right]\|Y_k - X_k\|$ for all k large enough. Thus, *(ii)* and *(iii)* are proven.

From the last inequality and the condition $\|A\|^2 + \|\hat{X}^{-1}B\|^2 < 1$, it follows that the two sequences $\{X_k\}$ and $\{Y_k\}$ converge linearly to the same limit, i.e. $\hat{X} = \hat{Y}$ and Eq. (1) has a unique positive definite solution $X_+ = \hat{X}$. □

We note that the condition $\|A\|^2 + \|\hat{X}^{-1}B\|^2 < 1$ is better than $\|A\|^2 + \|B\|^2 < 1$.

Now, we consider a modification of the iterative method (5).

$$(9)\qquad \begin{cases} X_0 = M,\ \ Y_0 = N,\ \ V_0 \le Y_0^{-1}, \\ V_{k+1} = V_k(2I - Y_kV_k), \\ X_{k+1} = I + A^*X_kA + B^*V_{k+1}B, \quad k = 0,1,\ldots \\ Y_{k+1} = I + A^*Y_kA + B^*X_{k+1}^{-1}B. \end{cases}$$

Lemma 1. *[6, Lemma 3.2] Let C and P be Hermitian matrices of the same order and let $P > 0$. Then $CPC + P^{-1} \ge 2C$.*

Theorem 2. *The sequences $\{V_k\}$, $\{X_k\}$ and $\{Y_k\}$ generated by iterative method (9) have the following properties:*

(i) $X_k \le X_{k+1} \le Y_{k+1} \le Y_k$, $V_k \le V_{k+1} \le Y_k^{-1}$, $k = 0,1,\ldots$;

(ii) $\lim_{k\to\infty} X_k = \tilde{X} \le \tilde{Y} = \lim_{k\to\infty} Y_k$, $\lim_{k\to\infty} V_k = \tilde{Y}^{-1}$ *and* $X_+ \in [\tilde{X}, \tilde{Y}]$, *where X_+ is a positive definite solution of Eq. (1).*

(iii) $\|Y_{k+1}-X_{k+1}\| \le \|A\|^2\|Y_k-X_k\| + \|X_{k+1}^{-1}B\|\|Y_k^{-1}B\|\|Y_k-X_{k+1}\| + \|B\|^2\|Y_k^{-1}-V_{k+1}\|$, $k = 0,1,\ldots$;

(iv) for all $\varepsilon > 0$, we have $\|Y_{k+1}-X_{k+1}\| \le \left[\|A\|^2 + (\|\tilde{X}^{-1}B\| + \varepsilon)^2\right]\|Y_k - X_k\|$ for all k large enough.

Moreover, if $\|A\|^2 + \|\tilde{X}^{-1}B\|^2 < 1$, then $X_+ = \tilde{X} = \tilde{Y}$ is a unique positive definite solution of Eq. (1) and the sequences $\{X_k\}$ and $\{Y_k\}$ converge linearly to X_+.

Proof. We prove the statement (i) by induction. We have $X_0 = M \le N = Y_0$. We compute

$$\begin{aligned} V_1 &= V_0(V_0^{-1} + V_0^{-1} - Y_0)V_0 \ge V_0V_0^{-1}V_0 = V_0, \\ X_1 &= I + A^*MA + B^*V_1B \ge I + A^*MA = M = X_0, \\ Y_1 &= I + A^*NA + B^*X_1^{-1}B \le I + A^*NA + B^*M^{-1}B = Y_0. \end{aligned}$$

Moreover, by Lemma 1 we have $V_1 = 2V_0 - V_0Y_0V_0 \le Y_0^{-1}$. Hence

$$X_1 = I + A^*MA + B^*V_1B \le I + A^*MA + B^*N^{-1}B \le I + A^*NA + B^*M^{-1}B = Y_0$$

and

$$Y_1 - X_1 = A^*(N-M)A + B^*(X_1^{-1} - V_1)B \ge B^*(X_1^{-1} - Y_0^{-1})B \ge 0.$$

Therefore, $V_0 \le V_1 \le Y_0^{-1}$ and $X_0 \le X_1 \le Y_1 \le Y_0$.

Assume that $V_{k-1} \le V_k \le Y_{k-1}^{-1}$ and $X_{k-1} \le X_k \le Y_k \le Y_{k-1}$.

We have

$$\begin{aligned} V_{k+1} - V_k &= V_k(V_k^{-1} - Y_k)V_k \ge V_k(V_k^{-1} - Y_{k-1})V_k \ge 0, \\ X_{k+1} - X_k &= A^*(X_k - X_{k-1})A + B^*(V_{k+1} - V_k)B \ge 0, \end{aligned}$$

and

$$Y_{k+1} - Y_k = A^*(Y_k - Y_{k-1})A + B^*(X_{k+1}^{-1} - X_k^{-1})B \le 0.$$

Moreover, by Lemma 1 we have $V_{k+1} = 2V_k - V_kY_kV_k \le Y_k^{-1}$. Hence

$$Y_k - X_{k+1} = A^*(Y_{k-1} - X_k)A + B^*(X_k^{-1} - V_{k+1})B \ge B^*(X_k^{-1} - Y_k^{-1})B \ge 0$$

and

$$Y_{k+1} - X_{k+1} = A^*(Y_k - X_k)A + B^*(X_{k+1}^{-1} - V_{k+1})B \ge B^*(X_{k+1}^{-1} - Y_k^{-1})B \ge 0.$$

Therefore, $X_k \le X_{k+1} \le Y_{k+1} \le Y_k$ and $V_k \le V_{k+1} \le Y_k^{-1}$ for $k = 0, 1, \ldots$.

(ii): From (i) it follows that the limits $\lim_{k\to\infty} X_k$, $\lim_{k\to\infty} Y_k$, and $\lim_{k\to\infty} V_k$ exist, and $\lim_{k\to\infty} X_k \le \lim_{k\to\infty} Y_k$, $\lim_{k\to\infty} V_k \le (\lim_{k\to\infty} Y_k)^{-1}$. Let $\lim_{k\to\infty} X_k = \tilde{X}$, $\lim_{k\to\infty} Y_k = \tilde{Y}$ and $\lim_{k\to\infty} V_k = \tilde{V}$. Taking limits in method (9) yields $\tilde{V} = \tilde{V}\tilde{Y}\tilde{V}$, $\tilde{X} = I + A^*\tilde{X}A + B^*\tilde{V}B$, and $\tilde{Y} = I + A^*\tilde{Y}A + B^*\tilde{X}^{-1}B$. Thus, $\tilde{V} = \tilde{Y}^{-1}$ and $\bar{X} = I + A^*\tilde{X}A + B^*\tilde{Y}^{-1}B$.

By subtracting the equations in (9), we obtain

$$\begin{aligned} Y_{k+1} - X_{k+1} &= A^*(Y_k - X_k)A + B^*(X_{k+1}^{-1} - V_{k+1})B \\ &= A^*(Y_k - X_k)A + B^*(X_{k+1}^{-1} - Y_k^{-1})B + B^*(Y_k^{-1} - V_{k+1})B \qquad (10) \\ &= A^*(Y_k - X_k)A + B^*X_{k+1}^{-1}(Y_k - X_{k+1})Y_k^{-1}B + B^*(Y_k^{-1} - V_k)Y_k(Y_k^{-1} - V_k)B \\ &\le A^*(Y_k - X_k)A + B^*X_{k+1}^{-1}(Y_k - X_{k+1})X_{k+1}^{-1}B + B^*(Y_k^{-1} - V_k)Y_k(Y_k^{-1} - V_k)B. \qquad (11) \end{aligned}$$

Moreover, we have

$$B^*(Y_k^{-1} - V_k)B \le B^*(X_k^{-1} - V_k)B = Y_k - X_k - A^*(Y_{k-1} - X_{k-1})A \le Y_k - X_k. \qquad (12)$$

Using (10), we obtain

$$\begin{aligned}\|Y_{k+1}-X_{k+1}\| &\le \|A^*(Y_k-X_k)A\| + \|B^*X_{k+1}^{-1}(Y_k-X_{k+1})Y_k^{-1}B\| + \|B^*(Y_k^{-1}-V_{k+1})B\| \\ &\le \|A\|^2\|Y_k-X_k\| + \|X_{k+1}^{-1}B\|\|Y_k^{-1}B\|\|Y_k-X_{k+1}\| + \|B\|^2\|Y_k^{-1}-V_{k+1}\|\end{aligned}$$

Thus, *(iii)* is proven.

Using (11) and (12), we obtain

$$\begin{aligned}\|Y_{k+1}-X_{k+1}\| &\le \|A\|^2\|Y_k-X_k\| + \|X_{k+1}^{-1}B\|^2\|Y_k-X_{k+1}\| + \|Y_kB^{-1}\|\|(Y_k^{-1}-V_k)B\|\|Y_k-X_k\| \\ &\le \left[\|A\|^2 + \|X_{k+1}^{-1}B\|^2 + \|Y_kB^{-1}\|\|(Y_k^{-1}-V_k)B\|\right]\|Y_k-X_k\| \\ &\le \left[\|A\|^2 + \left(\|\tilde{X}^{-1}B\| + \|X_{k+1}^{-1}-\tilde{X}^{-1}\|\|B\|\right)^2\right]\|Y_k-X_k\| \\ &\quad + \|Y_kB^{-1}\|\|B\|\left(\|Y_k^{-1}-\tilde{Y}^{-1}\| + \|\tilde{Y}^{-1}-V_k\|\right)\|Y_k-X_k\| \\ &\le \left\{\|A\|^2 + \left[\|\tilde{X}^{-1}B\| + \|B\|\|X_{k+1}^{-1}-\tilde{X}^{-1}\|\right. \right. \\ &\quad \left.\left. + \frac{\|Y_kB^{-1}\|\|B\|\left(\|Y_k^{-1}-\tilde{Y}^{-1}\| + \|\tilde{Y}^{-1}-V_k\|\right)}{2\left(\|\tilde{X}^{-1}B\| + \|B\|\|X_{k+1}^{-1}-\tilde{X}^{-1}\|\right)}\right]^2\right\}\|Y_k-X_k\|\end{aligned}$$

Since $X_k^{-1}\to\tilde{X}^{-1}$, $Y_k^{-1}\to\tilde{Y}^{-1}$, and $V_k\to\tilde{Y}^{-1}$, then for all $\varepsilon>0$ there exists an index ν_ε such that for $k>\nu_\varepsilon$ the condition

$$\|B\|\|X_{k+1}^{-1}-\tilde{X}^{-1}\| + \frac{\|Y_kB^{-1}\|\|B\|\left(\|Y_k^{-1}-\tilde{Y}^{-1}\| + \|\tilde{Y}^{-1}-V_k\|\right)}{2\left(\|\tilde{X}^{-1}B\| + \|B\|\|X_{k+1}^{-1}-\tilde{X}^{-1}\|\right)} < \varepsilon$$

holds.

Hence $\|Y_{k+1}-X_{k+1}\| \le [\|A\|^2 + (\|\tilde{X}^{-1}B\|+\varepsilon)^2]\|Y_k-X_k\|$ for all k large enough. Thus, *(iv)* is proven. □

3 Numerical experiments

In this section, we present some numerical examples to illustrate the convergence behavior of the algorithms under consideration, with $X_0=M$, $Y_0=N$, and $Z_0=(M+N)/2$, where M and N are the unique solutions to the equations $X-A^*XA=I$ and $X-A^*XA=I+B^*M^{-1}B$, respectively. We obtain M and N by the function `dlyap`.

As practical stopping criterion we use $err_k \le tol = 10^{-10}$, where $err_k = \|Y_k-X_k\|_\infty$ for methods (2), (5), (7), and (9), and $err_k = \|Z_k-Z_{k-1}\|_\infty$ for method (4). The norm used here is $\|A\|_\infty = \max_i\sum_{j=1}^n |a_{ij}|$. For the examples under consideration, we present: k – the number of iterations, err_k, res_k, as $res_k = res\left(\frac{X_k+Y_k}{2}\right)$ for methods (2), (5), (7), and (9), $res_k = res(Z_k)$ for method (4), where $res(X) = \|I+A^*XA+B^*X^{-1}B-X\|_\infty$, and *time* – the execution time of the corresponding algorithm.

Our main aim in this section is to compare methods (2) and (5) with the corresponding modifications (7) and (9). The computational complexity of methods (2) and (5) is identical to that of their respective modifications (7) and (9).

Example 1. *[2] We consider Eq. (1) with matrix*

$$A = \frac{1}{56}\begin{pmatrix} 1 & 5 & 3 & 2 \\ -1 & -6 & 3 & 4 \\ -1 & 3 & 7 & 5 \\ 1 & 8 & 2 & 1 \end{pmatrix}, \quad B = \frac{1}{70}\begin{pmatrix} 7 & 9 & 6 & 8 \\ 7 & 5 & 8 & 3 \\ 9 & 8 & 6 & 7 \\ 11 & 5 & 9 & 3 \end{pmatrix}.$$

For Example 1, we have $\rho(A) \approx 0.158$, $\rho(Z_{10}^{-1}B) \approx 0.34$, $\|A\|^2 + \|B\|^2 \approx 0.214$, and $\|A\|^2 + \|X_{11}^{-1}B\|^2 \approx 0.171$. It can be seen that the sufficient conditions $\|A\|^2 + \|B\|^2 < 1$ and $\|A\|^2 + \|X_{10}^{-1}B\|^2 < 1$ for the uniqueness of the positive definite solution, are satisfied.

In Table 1 are given the results of the experiment with Example 1. Methods (7) and (9) require fewer iterations than their respective base methods (2) and (5).

Table 1: Numerical results of Example 1.

Method	k	$\|Y_k - X_k\|$	$res(X_k)$	time
(2)	11	$6.9841e-11$	$5.7187e-13$	4.8360e-04
(7)	8	$5.2148e-12$	$1.7537e-12$	3.3020e-04
(4)	10	$5.8898e-11$	$5.5299e-12$	2.3780e-04
(5)	11	$7.5469e-11$	$4.1612e-13$	4.8520e-04
(9)	8	$6.0009e-12$	$2.0182e-12$	3.2980e-04

Example 2. *[5] We consider Eq. (1) with matrix coefficients*

$$A = \frac{1}{200}\begin{pmatrix} 41 & 15 & 23 & 35 & 66 \\ 25 & 12 & 27 & 45 & 21 \\ 23 & 27 & 28 & 16 & 24 \\ 15 & 45 & 16 & 52 & 65 \\ 66 & 21 & 24 & 65 & 35 \end{pmatrix}, \quad B = \frac{1}{30}\begin{pmatrix} 23 & 21 & 23 & 25 & 32 \\ 21 & 45 & 60 & 42 & 33 \\ 23 & 24 & 34 & 18 & 17 \\ 13 & 42 & 18 & 44 & 30 \\ 32 & 33 & 26 & 30 & 26 \end{pmatrix}.$$

For Example 2, we have $\rho(A) \approx 0.88$, $\rho(Z_{79}^{-1}B) \approx 0.47$, $\|A\|^2 + \|B\|^2 \approx 26.39$, and $\|A\|^2 + \|X_{169}^{-1}B\|^2 \approx 2.93$. Although $\|A\|^2 + \|X_{10}^{-1}B\|^2$ is significantly smaller than $\|A\|^2 + \|B\|^2$, it turns out to be larger than 1. Consequently, the sufficient conditions for the uniqueness of the solution are not met, but the methods under consideration converge to such a solution.

In Table 2 are given the results of the experiment with Example 2. Methods (7) and (9) require fewer iterations than their respective base methods (2) and (5). However, the relative reduction in iterations is smaller compared to Example 1.

Table 2: Numerical results of Example 1.

Methods	k	$\|Y_k - X_k\|$	$res(X_k)$	time
(2)	169	$9.4182e-11$	$2.6645e-15$	0.0060
(7)	160	$9.1872e-11$	$5.8131e-13$	0.0053
(4)	73	$9.7332e-11$	$7.0567e-11$	0.0014
(5)	172	$8.5978e-11$	$3.1086e-15$	0.0055
(9)	162	$9.9574e-11$	$6.3283e-13$	0.0051

Example 3. *We consider Eq. (1) with matrix coefficients*

$$A = \frac{9}{10}K + \frac{1}{130}K^T \text{ and } B = \frac{1}{200}F + \frac{1}{170}F^T,$$

where K and F are 50×50 matrices defined by

`gallery('kahan',50)` *and* `gallery('frank',50)`,

respectively.

For Example 3, we have $\rho(A) \approx 0.985$, $\rho(Z_{945}^{-1}B) \approx 0.819$, $\|A\|^2 + \|B\|^2 \approx 69.03$, and $\|A\|^2 + \|X_{1131}^{-1}B\|^2 \approx 31.72$.

Although the sufficient condition $\|A\|^2 + \|\hat{X}^{-1}B\|^2 < 1$ for the uniqueness of the solution for Example 3 is not met, the methods under consideration converge to such a solution. In Table 3 are given the results of the experiment. For this example, the number of iterations for methods (7) and (9) differ slightly from those for the corresponding methods (2) and (5). This is observed as $\rho(A)$ increases and approaches 1.

Table 3: Numerical results of Example 3.

Method	k	$\|Y_k - X_k\|$	$res(X_k)$	time
(2)	1131	$9.9941e-11$	$3.8867e-12$	0.8188
(7)	1129	$9.8282e-11$	$4.0714e-12$	0.7481
(4)	945	$9.8368e-11$	$1.0171e-10$	0.3192
(5)	1158	$9.8105e-11$	$3.7090e-12$	0.6452
(9)	1150	$9.9195e-11$	$3.0678e-12$	0.6373

Example 4. *We consider Eq. (1) with matrix coefficients*

$$A = \frac{1}{2000}F + \frac{1}{2600}F^T \quad \textit{and} \quad B = 30K + K^T,$$

where K and F are the matrices from Example 3.

For Example 4, we have $\rho(A) \approx 0.5$, $\rho(Z_{945}^{-1}B) \approx 0.93$, $\|A\|^2 + \|B\|^2 \approx 3.4e+04$, and $\|A\|^2 + \|X_{1131}^{-1}B\|^2 \approx 958.7$. In this example, the sufficient conditions for the uniqueness of a positive definite solution are again not met, nevertheless, the methods converge to such a solution.

In Table 4 are given the results of the experiment. The number of iterations for methods (7) and (9) is approximately half that of the corresponding methods (2) and (5). This is observed when $\rho(X_+^{-1}B)$ increases and approaches 1.

Table 4: Numerical results of Example 4.

Method	k	$\|Y_k - X_k\|$	$res(X_k)$	time
(2)	359	$8.9026e-11$	$3.3562e-11$	0.1142
(7)	184	$9.9842e-11$	$2.8233e-11$	0.0563
(4)	346	$9.7796e-11$	$7.9176e-11$	0.0593
(5)	370	$7.9336e-11$	$4.1817e-11$	0.1016
(9)	196	$6.9555e-11$	$2.0950e-11$	0.0506

Conclusion

Two iterative methods for finding a positive definite solution of the matrix equation $X - A^*XA - B^*X^{-1}B = I$ are considered. The sufficient condition $\|A\|^2 + \|B\|^2 < 1$ in [1] for the uniqueness of the positive definite solution has been improved. We obtain the condition $\|A\|^2 + \|\hat{X}^{-1}B\|^2 < 1$, where $I \leq \hat{X} = \lim_{k\to\infty} X_k$, and $\{X_k\}$ is the sequence, defined with considered methods. Theorems on the convergence rate of the considered methods are proven. The numerical

examples 2, 3 and 3 showed that the obtained conditions is not necessarily. Moreover, experiments show that the new modifications have an advantage in the case of increasing $\rho(X_+^{-1}B)$ and approaching 1.

REFERENCES:

[1] Ali, A., For the matrix equation $X - A^*XA - B^*X^{-1}B = I$. In: MATTEX 2018, Conference proceeding, vol. 1, Shumen, Bulgaria, (2018) 161–166 (in Bulgarian).

[2] Ali, A., Hasanov, V., An iterative method for solving the matrix equatin $X - A^*XA - B^*X^{-1}B = I$. Mathematical and Software Engineering, **6** (2020), 1–6.

[3] Ferrante, A., Levy, B., Hermitian solutions of the equation $X = Q + NX^{-1}N^*$. Linear Algebra Appl., **247** (1996) 359–373.

[4] Gao, D., Iterative methods for solving the nonlinear matrix equation $X - A^*X^pA - B^*X^{-q}B = I$ $(0 < p,q < 1)$. Advances in Linear Algebra and Matrix Theory, **7** (2017), 72–78.

[5] Hasanov, V., Necessary and sufficient condition for the existence of a positive definite solution of a matrix equation. Annual of Konstantin Preslavsky University of Shumen, **XX C**, (2019), 13–19.

[6] Zhan, X.Z., Computing the extreme positive definite solutions of a matrix equation. SIAM J. Sci. Comput., **17** (1996) 632–645.

[7] Ran, A.C.M., Reurings, M.C.B., The symmetric linear matrix equation. Electron. J. Linear A., **9** (2002) 93–107.

[8] Hasanov, V., On positive definite solutions of the matrix equation $X - A^*XA - B^*X^{-1}B = I$. (submited)

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