IDENTIFICATION OF MARINE VESSEL DYNAMICS UNDER WAVE DISTURBANCES VIA HANKEL DMDc*

GYURHAN H. NEDZHIBOV

ABSTRACT: *This paper presents a data-driven system identification framework based on Hankel Dynamic Mode Decomposition with Control (Hankel DMDc) for modeling complex coupled 5-DOF ship maneuvering dynamics (surge u, sway v, roll rate p, yaw rate r, and roll angle ϕ) under persistent wave excitations. The proposed approach combines Hankel time-delay embedding and Improved Hankel DMDc framework. Evaluated on the benchmark DaRUS dataset, the identified surrogate model demonstrates superior reconstruction fidelity and multi-step predictive capability. The results indicate that Hankel DMDc provides a computationally efficient, noise-robust, and expressive linear surrogate model that successfully bridges nonlinear hydrodynamic simulations and real-time control.*

MSC*: 65P99, 37M10, 37L65*

KEYWORDS: *Dynamic mode decomposition with control, DMD method, DMDc method, Koopman operator, Equation-free.*

DOI: https://doi.org/10.46687/EGQP1116

1 Introduction

The growing demand for maritime automation, safety, and energy efficiency has significantly increased the interest in advanced control strategies for Autonomous Surface Vessels (ASVs). Precise trajectory tracking, dynamic positioning, and motion stabilization remain challenging tasks due to highly nonlinear hydrodynamic forces, strong cross-coupling between directional and rotational degrees of freedom (DOFs), and persistent environmental disturbances such as ocean waves and currents [1].

Traditional ship maneuvering control approaches are predominantly based on physics-derived hydrodynamic models, such as the Abkowitz cross-flow formulation [2] and the Maneuvering Modeling Group (MMG) model [3]. Although these models provide valuable physical insight, their development requires extensive experimental campaigns, including planar motion mechanism (PMM) tests, towing tank experiments, or computationally expensive Computational Fluid Dynamics (CFD) simulations.

In recent years, the rapid evolution of data-driven methods and modern sensor technologies has shifted significant attention toward black-box and surrogate modeling techniques for maritime applications [4]. Among them, Dynamic Mode Decomposition (DMD), introduced by Schmid [5], enables the extraction of spatiotemporal coherent structures and the construction of linear low-rank approximations directly from measurement data, see also [6, 7, 8]. Early applications demonstrated the efficacy of classical DMD for forecasting ship maneuvers and motion responses in wave environments [9, 10]. Subsequent efforts expanded these boundaries by pairing DMD with machine learning estimators [11] and developing High-Order DMD (HODMD) to model non-linear ship motion time series, as well as course-keeping dynamics under shallow-water wave perturbations [12]. A critical challenge in applying pure DMD-based representations to control problems is accounting for unobserved states and exogenous forces. To address this, recent milestones include model-free system identification of surface vessels using Hankel DMD with Control [13] and probabilistic adaptations such as Bayesian Hankel DMD for digital twinning applications [14]. Inspired

*By financial support of Shumen University under Grant RD-08-91/02.02.2026.

by these advances, the present work introduces a comprehensive data-driven identification framework based on Hankel DMDc, applied to the benchmark DaRUS ship dataset.

The remainder of this paper is organized as follows. Section 2 introduces theoretical background. Section 3 presents the problem formulation, data preprocessing and the Hankel DMDc methodology for for maritime applications. Section 4 discusses the identification results and validation. Section 5 concludes the paper.

2 Theoretical Background

We begin with a brief overview of the DMDc framework.

2.1 DMDc framework

Dynamic mode decomposition with control (DMDc) is an extension of the classical DMD method, developed to analyze multidimensional systems that are influenced by external inputs or disturbances [15]. Unlike standard DMD, which considers only the autonomous system dynamics, DMDc incorporates time-dependent control inputs, enabling it to distinguish between internal system behavior and input-driven effects. In [16, 17], improved and alternative approaches for implementing DMDc are presented. Here we will describe the Improved DMDc approach following the formulation in [16].

Consider a sequential set of data arranged in $n \times m+1$ matrix

$$D = [\mathbf{x}_0, \ldots, \mathbf{x}_m] \tag{1}$$

where n is the number of state variables and $m+1$ is the number of observations (*snapshots*). The data $\mathbf{x}_i$ could be from measurements, experiments, or simulations collected at the time t_i from a given dynamical system, and assume that the data are equispaced in time with a time step $\triangle t$. To proceed, we use an arrangement of the data set into two large data matrices

$$X = [\mathbf{x}_0, \ldots, \mathbf{x}_{m-1}] \quad \text{and} \quad Y = [\mathbf{x}_1, \ldots, \mathbf{x}_m]. \tag{2}$$

The time-varying inputs to the system are collected to form another data matrix

$$\Gamma = [\mathbf{u}_0, \ldots, \mathbf{u}_{m-1}], \tag{3}$$

where $\mathbf{u}_i \in \mathbf{R}^q$ and q denotes the number of input variables.

The goal of DMDc is to find a relationship between the future state $\mathbf{x}_{k+1}$, the current state $\mathbf{x}_k$, and the external control inputs $\mathbf{u}_k$, given by the following relation

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k, \tag{4}$$

where $A \in R^{n\times n}$ is called the DMD operator and $B \in R^{n\times q}$ is called the input matrix. The relation (4) can be rewritten in terms of the snapshot matrices

$$Y = AX + B\Gamma. \tag{5}$$

We assume that both operators A and B are unknown and the goal is to obtain their estimates.

The system description (5) can therefore be rewritten in an augmented form as

$$Y = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} X \\ \Gamma \end{bmatrix} = G\Omega, \tag{6}$$

where

$$G = [A\ B] \in \mathbf{R}^{n\times(n+q)} \text{ and } \Omega = \begin{bmatrix} X \\ \Gamma \end{bmatrix} \in \mathbf{R}^{(n+q)\times m}. \tag{7}$$

We seek a best-fit solution for the operator G. It can be approximated by

$$G = Y\Omega^{\dagger}, \tag{8}$$

where $\Omega^{\dagger}$ is the Moore-Penrose pseudoinverse of Ω. For this purpose we can use the singular value decomposition (SVD): $\Omega = U\Sigma V^{*}$. This expression may be truncated to the form

$$\Omega \approx \tilde{U}\tilde{\Sigma}\tilde{V}^{*}, \tag{9}$$

where $\tilde{U} \in \mathbf{R}^{(n+q)\times p}$, $\tilde{\Sigma} \in \mathbf{R}^{p\times p}$ and $\tilde{V} \in \mathbf{R}^{m\times p}$, which gives the approximation

$$G \approx Y\tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^{*}. \tag{10}$$

We can represent the matrix $\tilde{U}$ as follows

$$\tilde{U} = \begin{bmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{bmatrix}, \tag{11}$$

where $\tilde{U}_1 \in \mathbf{R}^{n\times p}$, $\tilde{U}_2 \in \mathbf{R}^{q\times p}$.

Now, let us represent the matrix $\Omega^{\dagger}$ as follows

$$\Omega^{\dagger} = \begin{bmatrix} \Omega_1 & \Omega_2 \end{bmatrix}, \tag{12}$$

where $\Omega_1 \in \mathbf{R}^{m\times n}$ and $\Omega_2 \in \mathbf{R}^{m\times q}$.

Then, for the matrix representation $\Omega^{\dagger}$, from (12), we get

$$\Omega^{\dagger} = \begin{bmatrix} \Omega_1 & \Omega_2 \end{bmatrix} = \begin{bmatrix} \tilde{V}\tilde{\Sigma}^{-1}\tilde{U_1}^{*} & \tilde{V}\tilde{\Sigma}^{-1}\tilde{U_2}^{*} \end{bmatrix}, \tag{13}$$

where $\Omega_1 \in \mathbf{R}^{m\times n}$, $\Omega_2 \in \mathbf{R}^{m\times q}$ and matrices $\tilde{U_1}$, $\tilde{U_2}$ are defined by (11). For the sake of convenience, let us denote

$$\Omega_1 = \tilde{V}\tilde{H}, \tag{14}$$

where $\tilde{H} = \tilde{\Sigma}^{-1}\tilde{U_1}^{*}$ is $p \times n$ matrix. We utilize reduced-order SVD of $\tilde{H}$

$$\tilde{H} \approx \bar{U}\check{\Sigma}\check{V}^{*}, \tag{15}$$

where $\bar{U} \in \mathbf{R}^{p\times r}$, $\check{\Sigma} \in \mathbf{R}^{r\times r}$ and $\check{V} \in \mathbf{R}^{m\times r}$ and r denotes the dimension of the subspace ($r \leq p$). Substituting (15) into (14), we get

$$\Omega_1 = \tilde{V}\bar{U}\check{\Sigma}\check{V}^{*} = \check{U}\check{\Sigma}\check{V}^{*}, \tag{16}$$

where $\check{U} = \tilde{V}\bar{U}$. Using the transformation matrix $\check{V}$, a low-dimensional representation of the matrices A and B can be computed as follows:

$$\begin{aligned} \tilde{A} &= \check{V}^{*}Y\check{U}\check{\Sigma}, \\ \tilde{B} &= \check{V}^{*}Y\Omega_2, \end{aligned} \tag{17}$$

where $\tilde{A} \in \mathbf{R}^{r\times r}$ and $\tilde{B} \in \mathbf{R}^{r\times q}$.

2.2 Hankel Time-Delay Embedding

Let h denote the embedding dimension. The augmented (lifted) state is defined as:

$$\mathbf{x}_k^{(H)} = \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{k-1} \\ \vdots \\ \mathbf{x}_{k-h+1} \end{bmatrix} \in \mathbb{R}^{nh}. \tag{18}$$

Similarly, the input vectors are augmented:

$$\mathbf{u}_k^{(H)} = \begin{bmatrix} \mathbf{u}_k \\ \mathbf{u}_{k-1} \\ \vdots \\ \mathbf{u}_{k-h+1} \end{bmatrix}. \tag{19}$$

The resulting lifted linear system becomes:

$$\mathbf{x}_{k+1}^{(H)} \approx \mathbf{A}_H \mathbf{x}_k^{(H)} + \mathbf{B}_H \mathbf{u}_k^{(H)}. \tag{20}$$

Using (18) and (18) the augmented snapshot matrices are constructed:

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1^{(H)} & \mathbf{x}_2^{(H)} & \cdots & \mathbf{x}_{N-1}^{(H)} \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} \mathbf{x}_2^{(H)} & \mathbf{x}_3^{(H)} & \cdots & \mathbf{x}_N^{(H)} \end{bmatrix} \tag{21}$$

and the corresponding input matrix is:

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1^{(H)} & \mathbf{u}_2^{(H)} & \cdots & \mathbf{u}_{N-1}^{(H)} \end{bmatrix}. \tag{22}$$

The relation (20) can be rewritten in terms of the augmented snapshot matrices

$$\mathbf{Y} = \mathbf{A}_H \mathbf{X} + \mathbf{B}_H \mathbf{U}. \tag{23}$$

which corresponds to (4) and therefore we can apply the DMDc approach, described in Section 2.1, to approximate $\mathbf{A}_H$ and $\mathbf{B}_H$.

3 Problem Formulation and Methodology

This section describes the proposed data-driven framework for extracting maneuvering trends from ship motion signals. The method combines Hankel time-delay embedding and Dynamic Mode Decomposition with control to construct a representation of the measured dynamics.

3.1 Nonlinear Ship Dynamics

The motion of a surface vessel can be described by a nonlinear dynamical system of the form:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t)), \tag{24}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\mathbf{u}(t) \in \mathbb{R}^m$ represents control inputs, and $\mathbf{w}(t) \in \mathbb{R}^d$ denotes external disturbances.

In this work, a coupled 5-DOF representation is considered:

$$\mathbf{x} = [u, v, p, r, \phi]^\top, \tag{25}$$

where u and v are surge and sway velocities, p and r are roll and yaw rates, and ϕ is the roll angle.

The control input vector is defined as:

$$\mathbf{u} = [\delta_l, \delta_r, n]^\top, \tag{26}$$

where δ_l and δ_r are the rudder angles, and n is the propeller shaft speed.

Environmental disturbances (primarily wave-induced effects) are modeled as:

$$\mathbf{w} = [V_w, \alpha_x, \alpha_y]^\top. \tag{27}$$

Due to strong nonlinearities, cross-coupling effects, and uncertain hydrodynamic parameters, obtaining an accurate analytical model $\mathbf{f}(\cdot)$ is challenging.

Given time-series measurements, the system is represented in discrete time:

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k), \tag{28}$$

where k denotes the discrete time index.

The objective is to approximate this nonlinear mapping using a linear model in an augmented lifted space (DMDc Formulation).

We seek a linear approximation of the form:

$$\mathbf{x}_{k+1} \approx \mathbf{A}\mathbf{x}_k + \mathbf{B}_u\mathbf{u}_k + \mathbf{B}_w\mathbf{w}_k, \tag{29}$$

where $\mathbf{A}$, $\mathbf{B}_u$, and $\mathbf{B}_w$ are system matrices to be identified from data.

To improve the representation of nonlinear dynamics and capture memory effects, a Hankel time-delay embedding is introduced, as introduced in Section 2.1. The resulting lifted linear system becomes:

$$\mathbf{x}^{(H)}_{k+1} \approx \mathbf{A}_H\mathbf{x}^{(H)}_k + \mathbf{B}_{u,H}\mathbf{u}^{(H)}_k + \mathbf{B}_{w,H}\mathbf{w}^{(H)}_k. \tag{30}$$

This relation can be rewritten in terms of the augmented snapshot matrices

$$\mathbf{Y} = \mathbf{A}_H\mathbf{X} + \mathbf{B}_{u,H}\mathbf{U} + \mathbf{B}_{w,H}\mathbf{W}, \tag{31}$$

where $\mathbf{Y}$, $\mathbf{X}$ and $\mathbf{U}$ are defined as in (21)-(21), and similarly for disturbances:

$$\mathbf{W} = \begin{bmatrix} \mathbf{w}^{(H)}_1 & \mathbf{w}^{(H)}_2 & \cdots & \mathbf{w}^{(H)}_{N-1} \end{bmatrix}. \tag{32}$$

Actually, equation (31) corresponds to (23), where $\mathbf{B}_H = [\mathbf{B}_{u,H}, \mathbf{B}_{w,H}]^T$.

3.2 Hankel DMDc Algorithm

We can generalize an algorithm for the model formulated in this way. The objective is to identify a reduced-order linear system of the form (31).

The algorithm consists of four main stages:

1. Data preprocessing and normalization
2. Hankel matrix construction
3. Low-rank approximation via SVD
4. System matrix identification

3.2.1 Data Preprocessing

The measured state, input, and disturbance data are first normalized using Z-score normalization:

$$\tilde{x} = \frac{x - \mu_x}{\sigma_x}, \tag{33}$$

where μ_x and σ_x denote the mean and standard deviation of each variable. This step ensures numerical conditioning, especially when variables have different physical scales (e.g., rad/s vs. rpm).

3.2.2 Hankel Matrix Construction

Given a sequence of measurements $\{\mathbf{x}_k\}_{k=1}^{N}$, a block Hankel matrix is constructed as:

$$\mathbf{X}_H = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_{N-h} \\ \mathbf{x}_2 & \mathbf{x}_3 & \cdots & \mathbf{x}_{N-h+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_h & \mathbf{x}_{h+1} & \cdots & \mathbf{x}_{N-1} \end{bmatrix}. \tag{34}$$

Similarly, Hankel matrices are constructed for inputs and disturbances:

$$\mathbf{U}_H, \quad \mathbf{W}_H. \tag{35}$$

The time-shifted matrix is defined as:

$$\mathbf{Y}_H = \begin{bmatrix} \mathbf{x}_2 & \mathbf{x}_3 & \cdots & \mathbf{x}_{N-h+1} \\ \mathbf{x}_3 & \mathbf{x}_4 & \cdots & \mathbf{x}_{N-h+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_{h+1} & \mathbf{x}_{h+2} & \cdots & \mathbf{x}_N \end{bmatrix}. \tag{36}$$

3.2.3 Augmented Data Matrix

The augmented data matrix is formed as:

$$\mathbf{\Omega} = \begin{bmatrix} \mathbf{X}_H \\ \mathbf{U}_H \\ \mathbf{W}_H \end{bmatrix}. \tag{37}$$

This matrix encapsulates both system states and exogenous inputs. Note that matrix $\mathbf{\Omega}$ corresponds to matrix $\Omega = [X, \Gamma]^T$ in (7) for $\Gamma = [\mathbf{U}_H, \mathbf{W}_H]^T$.

3.2.4 Low-Rank Approximation via SVD

According to (9), in order to reduce noise and computational complexity, Singular Value Decomposition is applied:

$$\mathbf{\Omega} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^* \tag{38}$$

and truncated approximation is then computed:

$$\mathbf{\Omega}_r = \mathbf{U}_r \mathbf{\Sigma}_r \mathbf{V}_r^*, \tag{39}$$

where r is selected such that:

$$\frac{\sum_{i=1}^{r} \sigma_i^2}{\sum_{i=1}^{\text{all}} \sigma_i^2} \geq \eta, \tag{40}$$

with η typically chosen as 0.995.

3.2.5 System Matrix Identification

Using relations (14)–(16) and (17), the matrices $\mathbf{A}_H$ and $\mathbf{B}_H$ are extracted as

$$\begin{aligned} \mathbf{A}_H &= \check{\mathbf{V}}^* \mathbf{Y} \check{\mathbf{U}} \check{\mathbf{\Sigma}}, \\ \mathbf{B}_H &= \check{\mathbf{V}}^* \mathbf{Y} \mathbf{\Omega}_2, \end{aligned} \tag{41}$$

where $\mathbf{B}_H = [\mathbf{B}_{u,H}, \mathbf{B}_{w,H}]^T$.

3.2.6 Algorithm Summary

The complete Hankel DMDc procedure is summarized below:

Algorithm 1 Hankel DMDc Identification

1: Collect time-series data $\mathbf{x}, \mathbf{u}, \mathbf{w}$
2: Normalize data using Z-score
3: Construct Hankel matrices $\mathbf{X}_H, \mathbf{U}_H, \mathbf{W}_H$
4: Form $\mathbf{\Omega} = [\mathbf{X}_H; \mathbf{U}_H; \mathbf{W}_H]$
5: Compute SVD: $\mathbf{\Omega} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$
6: Select rank r based on energy threshold
7: Compute $\mathbf{A}_H$ and $\mathbf{B}_H$ according to (41)
8: Extract $\mathbf{B}_{u,H}$ and $\mathbf{B}_{w,H}$
9: Compute lifted-state evolution: $\hat{\mathbf{Y}}_H = \mathbf{A}_H \mathbf{X}_H + \mathbf{B}_H \mathbf{U}_H + \mathbf{B}_{w,H} \mathbf{W}_H$
10: Reconstruct physical states via anti-diagonal averaging
11: Apply inverse Z-score transformation to recover physical variables

Despite the increased dimensionality due to Hankel embedding, the resulting model remains linear and thus suitable for real-time prediction and control applications.

4 Results and Validation

This section evaluates the performance of the proposed Hankel DMDc framework in terms of reconstruction fidelity and predictive accuracy. The analysis is conducted on simulated ship maneuvering data subject to wave disturbances.

4.1 Experimental Setup

The dataset consists of multivariate time series describing vessel motion states:

$$\mathbf{x} = [u, v, p, r, \phi]^\top,$$

with control inputs:

$$\mathbf{u} = [\delta_l, \delta_r, n]^\top,$$

and wave disturbances:

$$\mathbf{w} = [V_w, \alpha_x, \alpha_y]^\top.$$

The sampling time is $T_s = 1$ s. The Hankel embedding dimension is set to $h = 40$, and the truncation rank is selected based on 99.5% energy retention. The model is trained using a sliding window of $N_{\text{train}} = 300$ samples.

4.2 Reconstruction Performance

To assess the ability of the model to recover the original signals, the reconstructed states $\hat{\mathbf{x}}_k$ are compared against the measured data. Fig. 1 shows the reconstruction of 5 sygnals.

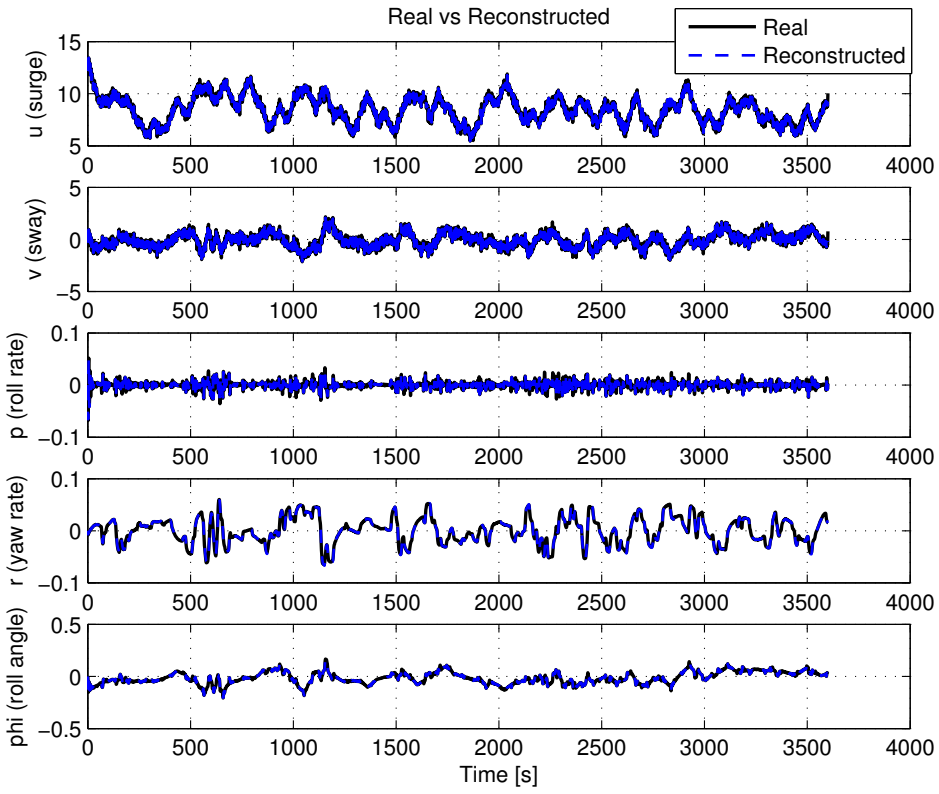


Figure 1: Reconstruction of 5 signals using Hankel DMDc.

The results demonstrate that the proposed method accurately captures both low-frequency maneuvering trends and higher-frequency oscillatory components induced by waves.

4.3 Error Metrics

Reconstruction accuracy is quantified using the Root Mean Square Error (RMSE):

$$\text{RMSE}_s = \sqrt{\frac{1}{N}\sum_{k=1}^{N}\left(x_{s,k} - \hat{x}_{s,k}\right)^2}. \tag{42}$$

Table 1 summarizes the reconstruction errors across all state variables.

State	RMSE
u (surge)	0.0585
v (sway)	0.0342
p (roll rate)	0.0006
r (yaw rate)	0.0010
ϕ (roll angle)	0.0005

Table 1: Reconstruction Error Metrics

5 Conclusion

This paper has presented a data-driven system identification framework based on Hankel DMDc applied to complex 5-DOF ship maneuvering dynamics under persistent wave excitations. By integrating time-delay embedding with Z-score standardization and energy-thresholded SVD truncation, the proposed method lifts the strongly nonlinear hydrodynamic state trajectory into a higher-dimensional linear subspace without requiring prior knowledge of hydrodynamic derivatives.

The results demonstrate that the Hankel DMDc approach significantly improves reconstruction accuracy compared to standard DMD and DMDc. This improvement is expected, as the Hankel embedding effectively reconstructs the underlying state space using time-delayed coordinates, in the spirit of Takens' embedding theorem. These findings demonstrate that Hankel DMDc serves as a computationally efficient, accurate, and noise-robust linear surrogate model.

REFERENCES:

[1] Fossen, T.I., *Handbook of Marine Craft Hydrodynamics and Motion Control*, 1st ed. Chichester, UK: John Wiley & Sons, (2011).

[2] Abkowitz, M.A., *Lectures on Ship Hydrodynamics?Steering and Maneuverability*, Hydro-and Aerodynamic Laboratory, Report No. Hy-5, Lyngby, Denmark, (1964).

[3] Yasukawa,H., Yoshimura,Y., *Introduction of MMG standard method for ship maneuvering predictions*, Journal of Marine Science and Technology, 20, (2015), 37–52.

[4] Cademartori, G., Oneto, L., Valdenazzi, F., Coraddu, A., Gambino, A., Anguita, D., A review on ship motions and quiescent periods prediction models, Ocean Engineering, Volume 280, (2023), 114822, ISSN 0029-8018,

[5] Schmid, P.J., *Dynamic mode decomposition of numerical and experimental data*, Journal of Fluid Mechanics, 656, (2010), 5–28.

[6] Kutz, J.N., Brunton, Bingni,S.L., Brunton,W. and Proctor, J., Dynamic Mode Decomposition: Data-Driven Modeling of Complex Systems, SIAM 2016, ISBN 978-1-611-97449-2, (2016), 1–234.

[7] Nedzhibov, G., On Alternative Algorithms for Computing Dynamic Mode Decomposition. Computation, (2022), 10, 210.

[8] Nedzhibov, G., Delay-Embedding Spatio-Temporal Dynamic Mode Decomposition. Mathematics, (2024), 12, 762.

[9] Diez, M., Serani,A., Campana, E.F., et al., Time-series forecasting of ships maneuvering in waves via dynamic mode decomposition. J. Ocean Eng. Mar. Energy 8, 471?478 (2022).

[10] Serani, A., Dragone, P., Stern, F., Diez, M., On the use of dynamic mode decomposition for time-series forecasting of ships operating in waves, Ocean Engineering, Volume 267, (2023), 113235, ISSN 0029-8018.

[11] Diez,M., Gaggero,M., and Serani, A., Data-driven forecasting of ship motions in waves using machine learning and dynamic mode decomposition, International Journal of Adaptive Control and Signal Processing 39, no. 10, (2025): 2119?2142.

[12] Chen, C., Yu, T., Zou, L., Zou, Z., Data-driven modeling and prediction of ship course keeping in shallow water waves using higher order dynamic mode decomposition incorporating control input,(2026), preprint https://doi.org/10.21203/rs.3.rs-8865149/v1

[13] Palma, G., Serani, A., Aram, S., Wundrow, D.W., Drazen, D., Diez, M., Model-free system identification of surface ships in waves via Hankel dynamic mode decomposition with control, Ocean Engineering, Volume 341, Part 3, (2025), 122539, ISSN 0029-8018.

[14] Palma, G., Serani, A., Aram, S., Wundrow, D.W., Drazen, D., Diez, M., Bayesian Hankel dynamic mode decomposition for ship motion digital twinning, Applied Ocean Research, Volume 165, (2025), 104863, ISSN 0141-1187.

[15] Proctor, J., Brunton, S.L., Kutz, J.N., *Dynamic mode decomposition with control*, SIAM Journal on Applied Dynamical Systems, 15(1), (2016), 142–161.

[16] Nedzhibov, G., An Improved Approach for Implementing Dynamic Mode Decomposition with Control, Computation, 11(10):201, (2023).

[17] Nedzhibov, G., An Alternative Framework for Dynamic Mode Decomposition with Control, AppliedMath, 5(2):60, (2025).

Authors' names: Gyurhan Nedzhibov
Affiliation: Faculty of Mathematics and Informatics, Shumen University, Shumen 9700, Bulgaria
ORCID ID: https://orcid.org/0000-0002-0474-1987
E-mail: g.nedzhibov@shu.bg