Annual of Konstantin Preslavsky University of Shumen
Faculty of Mathematics and Informatics
vol. XXVII C, 2026, pp. 3 – 8, https://doi.org/10.46687/ULQF7788

Some necessary conditions for the existence of a positive definite solution of the matrix equation $X + A^*X^{-2}A + B^*X^{-2}B = I^*$

AYNUR A. ALI, EMINE A. KARATASH, RADKA P. RUSSEVA

ABSTRACT: *In this paper we derive some necessary conditions for the matrix equation $X + A^*X^{-2}A + B^*X^{-2}B = I$ to have a Hermitian positive definite (HPD) solution. Relations between the solution X and the matrices A and B are proposed.*

KEYWORDS: *nonlinear matrix equation, positive definite solution, sufficient conditions*

2020 Math. Subject Classification:65F10; 65F15; 65F30; 15A24

1 Introduction

Consider the matrix equation

$$X + A^*X^{-2}A + B^*X^{-2}B = I, \tag{1}$$

where X is a unknown matrix, A and B are $n \times n$ complex matrices, and I is the $n \times n$ identity matrix. The equation (1) considered of Naglaa M. El-Shazly [1]. Necessary and sufficient conditions for the existence of a HPD solution have been obtained.

Specifically, if $B = 0$ the equation (1) reduced to

$$X + A^*X^{-2}A = I, \tag{2}$$

has been studied recently by several authors [4, 6]. There are necessary and sufficient conditions for the existence of a HPD solution. The iterative HPD solutions and the properties of the equation $X + A^*X^{-2}A = I$ have been discussed in [6].

*This work was partially supported by the Konstantin Preslavsky University of Shumen under project No RD-08-58/27.01.2026 and No RD-08-114/05.02.2026.

Hasanov and Ivanov in [5] considered the equation

$$X + A^* X^{-n} A = I. \tag{3}$$

Necessary and sufficient conditions for the nonlinear matrix equation (3) have been obtained. Some properties on the matrix A are presented. Two iterative processes for computing a HPD solution of the equation (3) are studied.

Zhang and Li [7] studied the HPD solution and presented Newton's method for the equation

$$X^s + A^* X^{-\alpha} A + B^* X^{-\beta} B = Q, \tag{4}$$

where Q is an $n \times n$ positive definite matrix equation, s, α and β are positive integers, whit $\alpha \geq 1$, $\beta \geq 1$.

Based on previous results, it can be seen that there are few works on the property and the iteration algorithm of the HPD solution for the nonlinear matrix equation (1). Motivated by above investigations [1, 5, 7], we study equation (1) and propose some necessary conditions for the existence of a positive definite solution.

Throughout this paper, we denote by A^* the conjugate transpose of A, $\rho(A)$ the spectral radius, by $\|A\|$ the spectral norm $\left(\|A\| = \sqrt{\rho(A^*A)}\right)$. $A > 0$ $(A \geq 0)$ means that A is a positive definite (semi-definite) matrix. If $A - B > 0$ (or $A - B \geq 0$) we write $A > B$ (or $A \geq B$). If A is $n \times n$ Hermitian matrix, $\lambda_{min}(A)$ and $\lambda_{max}(A)$ stand for the minimal and maximal eigenvalues of A. For $B \geq A > 0$ we use $[A, B]$ to denote the set of matrices $\{X : A \leq X \leq B\}$.

2 Preliminaries

In this section we will present some result obtained in [7]. Zhang and Li in [7] considered the equation (4). Necessary and sufficient conditions for the existence HPD solution have been obtained.

Theorem 1. *[7, Theorem 2.3] If equation (4) has a HPD solution, then*

$\overline{M} < X < \overline{N}$, *where*

$$\overline{M} = \frac{1}{2}[(AQ^{-1}A^*)^{\frac{1}{\alpha}} + (BQ^{-1}B^*)^{\frac{1}{\beta}}],$$

$$\overline{N} = \left[Q - \left(\frac{\lambda_n(Q)}{\lambda_1(Q)}\right)^{\frac{\alpha-1}{s}} A^*Q^{-\frac{\alpha}{s}}A - \left(\frac{\lambda_n(Q)}{\lambda_1(Q)}\right)^{\frac{\beta-1}{s}} B^*Q^{-\frac{\beta}{s}}B\right].$$

Remark 1. *For $\alpha = 2$, $\beta = 2$, $s = 1$ and $Q = I$, then $\overline{M} = \frac{1}{2}(\sqrt{AA^*} + \sqrt{BB^*})$ and $\overline{N} = I - A^*A - B^*B$.*

3 Main result

In this section, we propose some necessary existence conditions of the HPD solution for this equation (1).

Lemma 1. *[2] If $D \geq E > 0$ $(D > E > 0)$, then $D^q \geq E^q > 0$ $(D^q > E^q > 0)$ for all $q \in (0,1]$, and $E^q \geq D^q > 0$ $(E^q > D^q > 0)$ for all $q \in [-1,0)$.*

Lemma 2. *[3] If $A > B > 0$ and $uI \geq B \geq vI > 0$, then $(\frac{u}{v})^{\alpha-1}A^\alpha > B^\beta > 0$ for all $\alpha \in [1,+\infty)$.*

Theorem 2. *If equation (1) has an HPD solution X, then the following hold:*

(i) $X > \gamma\sqrt{AA^*} + (1-\gamma)\sqrt{BB^*}, \quad \gamma \in [0,1],$

(ii) $I - (A^*A + B^*B) - (\gamma\sqrt{AA^*} + (1-\gamma)\sqrt{BB^*}) > 0.$

Proof. (i) From equation (1), we get $A^*X^{-2}A < I$. Then $A^*X^{-1}X^{-1}A < I$, which is equivalent to

$$X^{-1}AA^*X^{-1} < I.$$

Hence $AA^* < X^2$. Using Lemma 1 we get

$$\sqrt{AA^*} < X. \tag{5}$$

Similarly, we have

$$\sqrt{BB^*} < X. \tag{6}$$

Therefore, combining (5) and (6), we obtained

$$\gamma\sqrt{AA^*} + (1-\gamma)\sqrt{BB^*} < X, \tag{7}$$

for all $\delta \in [0,1]$. (ii) Since I is identity matrix and hence by Lemma 2, we have $I \leq X^{-2}$. Using (i), we have:

$$\begin{aligned} 0 =& X + A^*X^{-2}A + B^*X^{-2}B - I \\ >& \gamma\sqrt{AA^*} + (1-\gamma)\sqrt{BB^*} + A^*X^{-2}A + B^*X^{-2}B - I \\ \geq& \gamma\sqrt{AA^*} + (1-\gamma)\sqrt{BB^*} + A^*A + B^*B - I. \end{aligned}$$

Hence

$$I - (A^*A + B^*B) - (\gamma\sqrt{AA^*} + (1-\gamma)\sqrt{BB^*}) > 0.$$

□

Remark 2. *The result (ii) for $\gamma = \frac{1}{2}$ is $X > \frac{\sqrt{AA^*}+\sqrt{BB^*}}{2}$, which follows from a Theorem 2.3 of [7].*

Theorem 3. *If X is a HPD solution of (1), then*

(i) $X < I - \mu(A^*M^{-2}A + B^*M^{-2}B)$,
where $\mu = \frac{\lambda_{min}(M)}{\lambda_{max}(M)}$ *and* $M = I - A^*A - B^*B$,

(ii) $I > \mu(A^*M^{-2}A + B^*M^{-2}B) + \gamma\sqrt{AA^*} + (1-\gamma)\sqrt{BB^*}$.

Proof. (i) Let $M = I - A^*A - B^*B$. By Theorem 2 $X < M$. Using Lemma1 and Lemma 2 we get $X^{-1} > M^{-1}$ and

$$\frac{\lambda_{min}(M^{-1})}{\lambda_{max}(M^{-1})}X^{-2} > M^{-2}.$$

Therefore $X^{-2} > \mu M^{-2}$. On the other hand, from (1) we get

$$\begin{aligned} X =& I - A^* X^{-2} A - B^* X^{-2} B \\ <& I - A^* \mu M^{-2} A - B^* \mu M^{-2} B \\ =& I - \mu (A^* M^{-2} A - B^* M^{-2} B). \end{aligned}$$

(ii) Since (i) we have $X^{-2} > \mu M^{-2}$. Similarly of (ii) from Theorem 2 we obtain:

$$\begin{aligned} 0 =& X + A^* X^{-2} A + B^* X^{-2} B - I \\ >& \gamma \sqrt{AA^*} + (1-\gamma)\sqrt{BB^*} + \mu (A^* M^{-2} A - B^* M^{-2} B) - I. \end{aligned}$$

Hence

$$I > \mu (A^* M^{-2} A + B^* M^{-2} B) + \gamma \sqrt{AA^*} + (1-\gamma)\sqrt{BB^*}.$$

□

Remark 3. *We note that* $\mu (A^* M^{-2} A + B^* M^{-2} B) > A^* A + B^* B$.

4 Conclusion

In this paper we introduced the nonlinear matrix equation (1). We presented some properties and preliminary same. We are derived some necessary conditions for the nonlinear matrix equation (1) to have a HPD solution X.

REFERENCES:

[1] El-Shazly N.M., Necessary and Sufficient Conditions for the Existence of a Positive Definite Solution for the Matrix Equation $X + A^* X^{-2} A + B^* X^{-2} B = I$, Internatinal Journal of Mathematics Trends and Technology, 52, 22-26, (2017).

[2] Parodi M., La localisation des valeurs caracteristiquesdes matrices et ses applications, Paris:Gautheir-Villars, (1959).

[3] Furuta T., Operator inequalities associated with Holder-McCarthy and Kantorovoch inequalities, J Inequal Appl., 2, 137-148, (1998).

[4] Ivanov I.G., Hasanov V.I., Minchev B.V., On the Matrix Equation $X \pm A^T X^{-2} A = I$, Linear Algebra Appl., 327, 27-44, (2001).

[5] Hasanov V.I., Ivanov I., Positive Definite Solutions of the Equation $X + A^* X^{-n} A = I$, L.Vulkov, J.Wasniewski, and P. Yalamov (Eds.): NAA 2000, LNCS 1988, 377–384, (2001).

[6] Ivanov I., El-Sayed S., Properties of Positive Definite Solution of the Equation $X + A^* X^{-2} A = I$, Linera Algebra And Appl., 279, 303-316, (1998).

[7] Zhang J., li Sh., On the Hermitian positive definite solution and Newton's method for a nonlinear matrix equation, Linear and Multilinear Algebra, 69, 2093-2114, (2019).

Aynur Abdulova Ali
Konstantin Preslavsky University of Shumen
a.ali@shu.bg

Emine Ahmed Karatash
Konstantin Preslavsky University of Shumen
e.karatash@shu.bg

Radka Peneva Russeva
Konstantin Preslavsky University of Shumen
russeva@shu.bg