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TECHNOLOGIES FOR PREPARING STUDENTS TO SOLVE OPTIMIZATION PROBLEMS AT INTERNATIONAL LOGICAL PROBLEM-SOLVING CHAMPIONSHIPS

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ABSTRACT. *The indicative list of key competences for Europe highlights, among the foremost, the following: "the ability to solve problems." A productive basis for developing this competence in students is problem solving in mathematics in general and optimization problems in particular. This article examines different approaches to understanding optimization problems. It also illustrates a technology (methodology) for preparing students to solve optimization problems using problem sets from the International Championships in Logical Problem Solving.*

KEYWORDS: *optimization problems, training technologies, solving mathematical problems.*

2010 Mathematics Subject Classification: 97M10.

1 Introduction

The ability of an individual to quickly find optimal solutions under any life circumstances is a challenge of our time. The current legislation regulating the educational process in general secondary education institutions of Ukraine overall, and in mathematics in particular, provides for the development of students' readiness to solve optimization problems only in grades 10–11. In contrast, European and global educational practice employs an early-intervention methodology to develop students' competences in solving optimization problems. Is such an approach advisable? How can we spark students' interest in solving such problems? On what problem material can an early-intervention methodology for developing competences in optimization problem solving be implemented? Our search for answers to these questions revealed a wide range of information. In view of this, in this article we will focus solely on the technology for preparing students to solve optimization problems at the International Championships in Logical Problem Solving.

Based on the results of our review, it can be stated that dissertation research on optimization problems is, for all practical purposes, exhaustively focused on solving them by means of computer technologies. These are applied optimization problems in the fields of finance, transport, medicine, and various branches of the economy. There is a considerable body of research oriented toward higher education, whose topics reflect the search for general approaches to solving certain classes and types of optimization problems. N. V. Dobrovolska, in the article “Methodology for Using Information Technologies in Solving Optimization Problems” [6], described linear-programming optimization problems for the following economic tasks: production planning, structural optimization, rational use of production capacities, and the transportation problem. As a result of solving these problems, profit, material and labor resources, time, etc., are maximized or minimized. The article by Nataliia Kuzmina, “The Main Types of Analysis of Optimization Problems with Computer Support” [8], examines certain methodological aspects of teaching the fundamentals of the theory of optimization methods. This article is addressed to students of pedagogical universities. The researcher classifies types of analysis of optimization problems, including the problem-posing stage and the stage of obtaining the optimal solution. However, the article considers examples of applying the proposed approaches only by means of information technologies. I. V. Nykyforchyn, in the article “Solving Optimization Problems with Economic Content in Mathematics Lessons” [12], notes that an analysis of publications on this topic shows that neither the issue of forming economic culture among general-education school students nor the aspects of preparing school teachers for this category of problems are sufficiently covered in contemporary pedagogical research. In fact, as the researcher asserts, optimization problems themselves are rarely encountered in the school mathematics curriculum. Out of necessity, the author proposes the systematic introduction of optimization problems of varying levels of complexity into school practice for solving mathematical problems. The more accessible of these can be offered to students at different stages of their mathematical education and can orient them toward a critical understanding and analysis of the solution, as well as toward the application of optimization methods [12]. T. V. Zaitseva, in the article “Integrated Lessons in the Study of Mathematics” [7], draws attention to the difficulties of solving optimization problems without information technologies. Nevertheless, she notes the necessity of including such problems in the mathematics curriculum of secondary school. The author points out that the mathematical model for optimization problems proposed in secondary school is presented using linear dependencies, which are understandable and accessible to upper-secondary students. Such educational activity offers a wide range of opportunities for conducting instructional research that involves not only solving problems but also posing them; it facilitates graphical and computational experiments on the basis of which the student arrives at formulating hypotheses regarding the regularities under study [7]. D. Ye. Bobyliev [1] presents his view of the place and role of optimization problems in specialized (profile) school, highlighting a general plan for solving such problems. He argues for the advisability of introducing a project-based course “Optimization Problems” at school. If introducing such an elective course is not possible, the author sees a way forward in creating mathematical models during mathematics lessons, and in informatics lessons—implementing the algorithms for solving these problems in practice using information computer technologies.

Purpose of this article – to explain the rationale and possibilities of an early-intervention methodology for developing students’ competences in solving optimization problems.

Technology (from the Greek τέχνη — art, craftsmanship, skill; λόγος — word, study) is a body of knowledge and information about the sequence of individual production operations in the course of manufacturing something [2]. It was from the engineering and technical sphere that this term was borrowed into pedagogical theory and practice. Today, the fixed expression “pedagogical technologies” is in common use. In essence, pedagogical technologies are intended to identify algorithms for optimizing the educational space, culminating in an evaluation of the methods chosen.

The organic integration of pedagogical technologies into the educational process structures the activity of the subject teacher in collaboration with the student. This approach ensures not only the delineation of goals, but also the formulation of algorithms for achieving them. This, in turn, provides the basis for predictability of the final outcome of the educational process. In line with the research of M. Clark, T. Sakamoto, and K. Chadwick, the primary functions of pedagogical technologies are the application of theoretical knowledge to the solution of practice-oriented tasks. In unison with V. L. Ortynskyi, we define the concept of “pedagogical technology” as a certain range of forms, methods, and modes of instruction that are used systematically in the educational process and that, for the most part, lead to achieving a predictable educational result within an acceptable margin of deviation [13]. “Instructional (teaching) technology” is a comparatively narrower concept. Its essence lies in finding the optimal algorithm for attaining a specified instructional goal.

At present, the organization of the educational process in mathematics in institutions of general secondary education in Ukraine is regulated by the Law of Ukraine “On Education” [4] and the Law of Ukraine “On Complete General Secondary Education” [5]. Mathematical competence is one of the ten key competences envisaged by the New Ukrainian School concept, since the teaching of mathematics makes a significant contribution to the development of each key competence.

For Ukraine, the trends and priority directions in the development of European and global education have never been a matter of indifference; however, for the most part, Ukrainian educators, given the specifics of domestic traditions, have charted their own path in the education of children. It should be noted that there is no single agreed list of key competences for the countries of the world, since such a list is determined by the position of the society in each individual country. The implementation of the international project “Definition and Selection of Key Competencies,” carried out by the national institutes of educational statistics of Switzerland and the USA, did not lead to the formation of a corresponding package of competences. Instead, the Council of Europe symposium “Key Competencies for Europe” identified and formulated an indicative list of key competences for Europe. Among the foremost is the ability to solve problems.

A fruitful foundation for developing a student's ability to recognize mathematics in real-life situations, to model scenarios, and to apply mathematical knowledge to optimally solve real-world problems is the solving of optimization problems. These belong to the most engaging mathematical problems, for many reasons. One of them is that such problems often model real-life needs and challenges. People have always sought to use material, financial, time, and labor resources in the most rational way and, for given production volumes, to reduce costs to the minimum (to minimize), or—given fixed resources to ensure the maximum output of products. Problems of this type are so-called optimization problems. General methods for solving them are based on the theory of the corresponding sections of mathematical analysis.

General formulation of an optimization problem. Let X – be a subset of R^n (i.e. X $\subset R^n$), $f(x)$ – be a real-valued function on R^n (i.e. $f: R^n \to R^1$). It is required to find a point $x^{\circ} \in X$, at which the function $f(x)$ attains its minimum value. The set X s called the feasible set (or feasible region), and the function $f(x)$ – is the objective function. The general optimization problem is briefly written as:

(1) $$f(x) \to \min_{x \in X}$$

Note that finding a maximizer of a function $\phi(x)$ is equivalent to finding a minimizer of $f(x) = -\phi(x)$, In other words, it makes no difference whether an optimization problem is formulated as a minimization or as a maximization.

Mathematical optimization (often simply *optimization*) is, with respect to a certain criterion, the selection of the best option from a set of available alternatives. Thus, the essence of optimization lies in finding the “best” possible value of an objective function within its domain, encompassing various types of objective functions and various types of domains.

Solving optimization problems in the school mathematics curriculum is mainly (over 90%) based on students’ mastery of the basics of mathematical analysis and is largely reduced to algebraic and geometric problems of finding the maximum or minimum of an objective function. Our understanding and view of the place and role of optimization problems in the system of developing students’ research competence is as follows. We consider practice in solving optimization problems as a didactic determinant in forming students’ research competence. The range of methods used is characterized by exceptional diversity and breadth. It is advisable to begin acquainting students with optimization problems as early as possible, ensuring a propaedeutic approach in instruction. Then, on the basis of continuity, the work should be continued from grade five through eleven. The problem sets offered at various stages of the International Championships in Logical Problem Solving (hereafter in the text — the Championship) can serve as an illustration of a technology for preparing students to solve optimization problems. It should be noted that one of the features of the Championships is the large number of problems. For example, over 3 hours, students in grades 10–11 are offered 16 tasks to solve.

For the youngest school-aged participants of the Championship, an optimization problem usually takes the form of a story problem formulated as a challenge. Naturally, for this age category only single-criterion optimization problems are proposed. Involving students in international mathematical

Championships can serve as a fruitful foundation for developing a sustained interest in solving optimization problems.

Problem 1 (Grade 5). Grandma baked four pancakes of different sizes and placed them on a plate (see the illustration in the figure). Before bringing them to the table, Marichka wants to arrange the pancakes in order from the largest on the bottom to the smallest on the top (see the illustration in the figure). To do this, she has a wooden spatula. The girl slides her spatula under one of the pancakes (except for the very top one) and flips the entire stack of two, three, or four pancakes on the spatula. What is the smallest number of flips Marichka can use to place the pancakes in the indicated order?

Figure 1.

Solution**:**

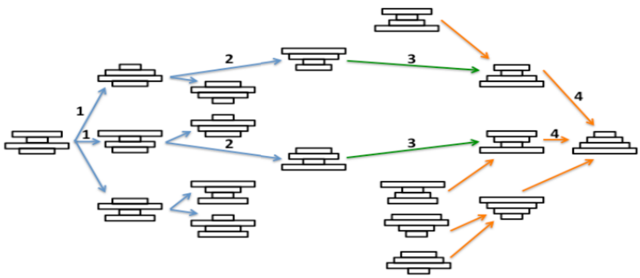

Figure 2.

Answer: 4.

It is clear that a technology for preparing students to solve such problems presupposes that younger schoolchildren, at an intuitive level, form the concepts of an objective function, feasible solutions, and a system of constraints. The solution to Problem 1 is presented in the form of a tree-like graph. Let us note that mathematics curricula in many European countries envisage introducing students to graph theory already in primary school. Graphs are widely used to solve various problems with the aid of computers and, in particular, to solve optimization problems. In an evident way (even for the youngest pupils), the problem of finding a path in a graph—including the shortest one—is solved on a graph. For unweighted graphs, the shortest path is the path consisting of the smallest (minimum) number of edges. For weighted graphs, it is the path with the smallest (minimum) total edge length.

The student-preparation technology for the Championships that we implement in our instructional practice requires younger schoolchildren to master this conceptual apparatus. In the course of such preparation, students learn the following methods for finding a path in an unweighted (as in Problem 1) graph: (1) exhaustive search of all options (ensuring completeness of the solution); (2) depth-first search (DFS); (3) breadth-first search (BFS). The solution method proposed above for an optimization problem clearly illustrates the basics of mathematical modeling and implicitly represents the application of the classical analytic–synthetic method. Undoubtedly, such a solution method eliminates difficulties related to analyzing the answer (the results obtained). Let us also note that students in grades 5–6, for the most part, interpret the notion of “optimal” as “the best, one that cannot be improved.” Students in this age group most often use the method of complete enumeration and the extremal principle to solve optimization problems.

<u>Problem 2 (Grade 8).</u> Find the value(s) of a for which the area of the triangle with side lengths $\sqrt{a^2 - a + 1}$, $\sqrt{a^2 + a + 1}$,

$\sqrt{4a^2+3}$ (assuming such a triangle exists) is maximal, and the value(s) of $a$for which the area is minimal.

Solution. It is easy to see that all real numbers are admissible values of the parameter a. We will prove the existence of a triangle with the side lengths given in the problem statement. Introduce a coordinate system on the plane$(L_1L_2L_3)$. Let O be the origin and choose the angles between $OL_1$and OL_2 and between OL_2 and OL_3 to be 60°. On the ray OL_3 lay off the segment $OC=a$, and on the ray OL_1 ay off the segment OA=1. By the law of cosines for ΔCOA we have: $CA=\sqrt{a^2+1-2a\cos 120°}=\sqrt{a^2+a+1}$. Next, lay off on the ray OL_2 the segment $OB=1$.

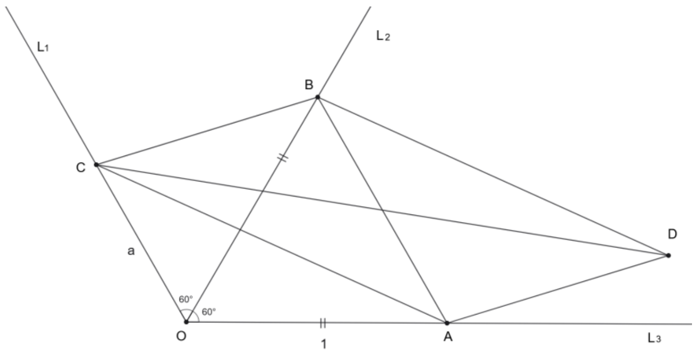

Figure 3.

By the law of cosines for ΔCOB we have:

(2) $$CB=\sqrt{a^2+1-2a\cos 60°}=\sqrt{a^2-a+1}$$

Similarly, $BA=1$. hrough point Bdraw a line parallel to line CA, and through point Adraw a line parallel to line CB. Let Dbe the intersection point of these lines; then $ACBD$is a parallelogram (by definition). Hence, the sum of the squares of its diagonals equals the sum of the squares of all its sides, i.e.,

(3) $$CD^2 + BA^2 = 2AC^2 + 2BC^2;$$

(4) $$CD = \sqrt{2AC^2 + 2BC^2 - BA^2}.$$

Substituting the previously found segment lengths, we obtain $CD = \sqrt{4a^2 + 3}$.

The above constitutes a constructive proof of the existence of a triangle with the side lengths given in the problem statement for any real a. In our case, this is ΔCAD.

(5) $$S_{\Delta CAD} = S_{\Delta ABC} = \frac{1}{2} S_{ACBD}.$$

Find $S_{\Delta ABC}$.

(6) $$S_{\Delta ABC} = S_{OCBA} - S_{\Delta COA} = S_{\Delta COB} + S_{\Delta OBA} - S_{\Delta COA} = \frac{\sqrt{3}}{4}a + \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}a = \frac{\sqrt{3}}{4}$$

Therefore $S_{\Delta CAD} = \frac{\sqrt{3}}{4}$ i.e., the area of the triangle in question is constant, and hence neither a maximum nor a minimum value exists.

Answer: $S_{\Delta CAD} = constant = \frac{\sqrt{3}}{4}$ sq. units

Note that in the course of solving this optimization problem, an analytic–constructive investigation established the existence of the triangle and identified the possible value of its area. By introducing an auxiliary element (a specific oblique coordinate system), the problem was modeled and solved by means of the coordinate method. The foregoing problem can model a practice-oriented, real-life situation in which one is required to minimize costs that, under the prevailing circumstances, are constant, or to maximize profits that, again under the circumstances, are constant.

Sometimes, to rationalize the solution of an algebraic optimization problem, it is advisable to reduce it to an equivalent geometric optimization problem (i.e., perform an equivalent transformation of the condition). This is especially appropriate when students have not yet mastered the basics of mathematical analysis, or when applying the derivative leads to equations of high complexity. We illustrate this approach with the following problem.

Problem 3 (Grade 9). Find the minimum value of the function

(7)
$$y = \sqrt{2x^2 - 2x + 1} + \sqrt{2x^2 - \left(\sqrt{3} - 1\right)x + 1} + \sqrt{2x^2 + \left(\sqrt{3} + 1\right)x + 1}$$

.

Solution.

(8)
$$y = \sqrt{2x^2 - 2x + 1} + \sqrt{2x^2 - \left(\sqrt{3} - 1\right)x + 1} + \sqrt{2x^2 + \left(\sqrt{3} + 1\right)x + 1} = \sqrt{x^2 + (x^2 - 2x + 1)} + \sqrt{\left(x^2 + x + \frac{1}{4}\right) + (x^2 - \sqrt{3}x + \frac{3}{4})} + \sqrt{\left(x^2 + x + \frac{1}{4}\right) + (x^2 + \sqrt{3}x + \frac{3}{4})} = \sqrt{x^2 + (x - 1)^2} + \sqrt{(x + \frac{1}{2})^2 + (x - \frac{\sqrt{3}}{2})^2} + \sqrt{(x + \frac{1}{2})^2 + (x + \frac{\sqrt{3}}{2})^2}$$

.

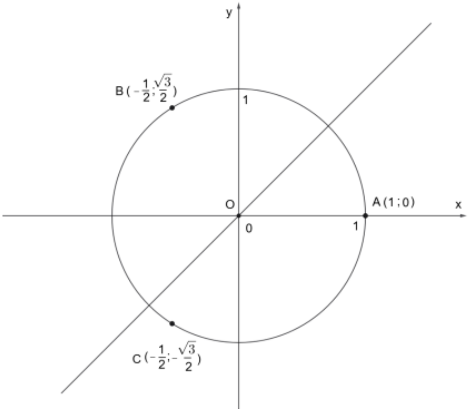

Figure 4.

Consider the unit circle and the line $y = x$. On this circle, mark three points with the following coordinates: $A(1;0), B\left(-\frac{1}{2};\frac{\sqrt{3}}{2}\right), C\left(-\frac{1}{2};-\frac{\sqrt{3}}{2}\right)$. Let $X(x;x)$ be some point on the line $y = x$; then

(9) $$AX = \sqrt{(x-1)^2 + x^2},$$

(10) $$BX = \sqrt{(x+\frac{1}{2})^2 + (x-\frac{\sqrt{3}}{2})^2},\ CX = \sqrt{(x+\frac{1}{2})^2 + (x+\frac{\sqrt{3}}{2})^2}.$$

Let us consider ΔABC. We note that the given problem reduces to the following equivalent geometric optimization problem: “On the line $y = x$ find a point for which the sum of its distances to the vertices of ΔABC the aforementioned triangle) is minimal.”

To solve this, we use the theorem on the property of the first Torricelli point. In a triangle whose largest angle is less than $120°$, the first Torricelli point has the minimal sum of distances to the triangle's vertices. (By definition, the first Torricelli point is a point from which all sides of the triangle are seen under an angle of $120°$. Such a point exists only in a triangle whose largest angle is less than $120°$, and for a given triangle this point is unique.)

Computing the side lengths of ΔABC, we obtain: $AB = BC = AC = \sqrt{3}$, therefore ΔABC s equilateral, and hence from point O – the center of its circumcircle, all sides of this triangle are seen under an angle of $120°$ (all the conditions of Torricelli's theorem are satisfied). t follows that the desired minimal sum of distances is numerically equal to $OA + OB + OC = 1 + 1 + 1 = 3$ (it is easy to see that $OA = OB = OC = 1 = R$ – the circumradius). Consequently, we finally have that the function

(11)

$$y = \sqrt{2x^2 - 2x + 1} + \sqrt{2x^2 - (\sqrt{3} - 1)x + 1} + \sqrt{2x^2 + (\sqrt{3} + 1)x + 1}$$

may attain its minimum value, which is numerically equal to 3.

Answer: 3.

Let us note that if this problem is approached by means of mathematical analysis, the student, in the course of a top-down analysis, will realize the inefficiency of such a method. It becomes necessary to model the problem and apply the method of equivalent transformation of the condition. The student searches for an equivalent geometric optimization problem and then studies the geometric configuration for compliance with the conditions of Torricelli’s theorem. The proposed solution to Problem 3 serves as an illustration of the organic combination of top-down analysis and bottom-up synthesis.

2 Conclusions

In view of European and global educational practice and based on our own experience of more than twenty years participating in the International Championships in Logical Problem Solving as a competitor, we can state that over 40% of the tasks in these competitions are optimization problems. About one third of all geometric problems in these competitions are, again, optimization problems. Such problems are offered to participants of all age categories, from first graders to people with higher mathematical education. Our European and global colleagues do not wait for the moment when students are introduced to derivatives and their applications. Instead, they begin teaching students to solve optimization problems from an early school age. We will not delve into a detailed justification of the soundness of this approach. We will simply note that the realities of life constantly demand of a person the ability to make optimal decisions quickly. Often the stakes are a human life; in the most critical situations, the stakes are the existence of a particular state or of humanity as a whole (the Chornobyl tragedy, war, etc.). Therefore, in our view, an early-intervention methodology for solving optimization problems is justified.

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